Covering Folded Shapes

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Abstract

Can folding a piece of paper flat make it larger? We explore whether a shape S must be scaled to cover a flat-folded copy of itself. We consider both single folds and arbitrary folds (continuous piecewise isometries $S \to \mathbb{R}^2$). The underlying problem is motivated by computational origami, and is related to other covering and fixturing problems, such as Lebesgue's universal cover problem and force closure grasps. In addition to considering special shapes (squares, equilateral triangles, polygons and disks), we give upper and lower bounds on scale factors for single folds of convex objects and arbitrary folds of simply connected objects.

1 Introduction

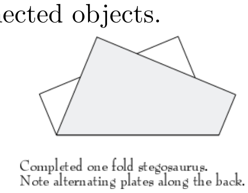

Figure 1: From Wu's diagram.

In this paper, we consider covering all possible folded versions of a given shape by a scaled copy of the shape itself, with the objective of keeping the scale factor as small as possible. We explore how folds can make an origami model larger, in the sense that Joseph Wu's one-fold stegosaurus[1] cannot be covered by a copy of the square from which it is folded.

Problems of covering a family of shapes by one minimum-cost object have a long tradition in geometry. The classical prototype is Lebesgue's universal cover problem from 1914 [10], which asks for a planar convex set of minimum area that can cover any planar set of diameter at most 1; Brass and Sharifi [3] give the best upper and lower bounds, but a gap remains. A similar question, also with a gap, is Moser's worm problem [9, 11], which asks for a convex set of minimum area that can cover any planar curve of length 1. As reported in [3] and the book by Brass, Moser, and Pach [2, Chapter 11.4], there is a large family of well-studied, but notoriously difficult problems parameterized by

- the family of sets to be covered,
- the sets allowed as covers,
- the size measure to be minimized, and
- the allowed transformations.

In this paper we consider a given *shape* S, which is a region of the plane that is a simply connected (no holes) closed 2-manifold with boundary (every interior point has a disk neighborhood and every boundary point a half-disk). A shape S may possess more specific properties: e.g., it may be convex, a (convex or non-convex) polygon, a disk, a square, or an equilateral triangle.

We denote by cS, for $c > 0$, the family of copies of S that have been scaled by c, and then rotated, reflected, and translated. We consider upper and lower bounds on the smallest constant c such that, for any F obtained by folding S, some member of cS contains or *covers* F. Let us be more specific about folding.

A *single fold of* S with line ℓ reflects one or more connected components of the difference $S \setminus \ell$ across ℓ. Let $\mathcal{F}_1(S)$ denote the family of shapes that can be generated by a single fold of S. An *arbitrary fold of* S is a continuous, piecewise isometry from $S \to \mathbb{R}^2$, which partitions S into a finite number of polygons and maps each rigidly to the plane so that the images of shared boundary points agree. The key property that we will use is that the length of any path in S equals the length of its image in $\mathbb{R}^2$. Let $\mathcal{F}(S)$ denote the family of all images of arbitrary folds of S.

The single fold and arbitrary fold are two simple notions of flat folding that avoid concerns of layering and fold order. Note that any upper bound that we prove for arbitrary folds applies to single folds, too. And, although the image of an arbitrary fold need not be the result of single folds, our lower bounds happen to be

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[1] An origami joke. http://www.josephwu.com/Files/PDF/stegosaurus.pdf

limits of finite sequences of single folds. Our results apply to 3-d folded shapes if *covering* is understood to mean covering the orthogonal projection to the plane.

Throughout this paper, we consider *origami covers*:

Definition 1 *For a given shape S and $c > 0$, cS is an* origami cover *of S if any member of $\mathcal{F}(S)$ can be covered by some member of cS. The* origami cover factor *of S is the smallest such c, which may be ∞:*

$$c^*(S) = \inf\{c \mid cS \text{ is a origami cover of } S\}.$$

Analogously, $c_1 S$ is a 1-fold origami cover *of S if any member of $\mathcal{F}_1(S)$ can be covered by $c_1 S$; and*

$$c_1^*(S) = \inf\{c \mid cS \text{ is a 1-fold cover of } S\}.$$

Questions of whether folding can increase area or perimeter have been considered before. It is clear that folding a piece of paper introduces overlap, so area can only decrease. On the other hand, the perimeter of a rectangle or square can be greater in a folded than an unfolded state—known as Arnold's ruble note or the Margulis napkin problem [1, 7]. Folding techniques that increase perimeter, like rumpling and pleat-sinking, make very small but spiky models that are easily covered by the original paper shape, however.

Before we explore single folds in the next section, let us recall some common geometric parameters of a shape S and make one general observation.

For a given shape S, an *incircle*, C_r, is a circle of maximum radius (the *inradius* r) contained in S. Similarly, the *circumcircle*, C_R, is the circle of minimum radius (the *circumradius*) that contains S. For a non-convex shape S, we instead measure geodesic distances within S, i.e., the distance between two points is the length of a shortest path in S between the points. A *geodesic diameter* is a path within S that attains the maximum distance D between two points of S. A *geodesic center* is a point in S that minimizes the maximum distance (the *geodesic radius* R) to all points of S. For convex shapes the geodesic radius R is also the circumradius. Jung's theorem in the plane says $\sqrt{3}R \le D \le 2R$, with the equilateral triangle and circle giving the two extremes [12, ch. 16].

For any folded state of S, these parameters give an upper bound on the origami cover factor.

Lemma 2 *Any shape S with inradius r and geodesic radius R has an origami cover factor $c^*(S) \le R/r$.*

Proof. Place any folded state $F \in \mathcal{F}(S)$ in the plane so that the image of a geodesic center is at the origin. Choose a member of $(R/r)S$ with an incircle center at the origin. Because no path in F can be more than R from the origin, the scaled incircle covers F. □

2 Single Folds

In this section we explore the 1-fold cover factor $c_1^*(S)$, giving general bounds for convex S and for polygons, and the exact values for equilateral triangles, squares, and a family derived from disks.

2.1 Convex shapes

For a convex set S, there is a lower bound for the 1-fold cover factor $c_1^*(S)$ that is within a constant factor of the upper bound given by Lemma 2.

Theorem 3 *Let S be a convex shape with inradius r and circumradius R. Then $\kappa R/r \le c^*(S) \le R/r$ for an appropriate constant $\kappa = ((\sqrt{5}-1)/2)^{5/2} \approx 0.300283$.*

Proof. The upper bound is from Lemma 2.

For the lower bound, consider the center p^* of the R-circle C_R that contains S. Because R is smallest possible, the set of points where the boundary of C_R touches S, $T := \partial C_R \cap S$, must contain at least two points, and no open halfplane through p^* can contain all of T. If $|T| = 2$, then these two points t_1 and t_2 must lie on a diameter of C_R; if $|T| > 2$, there must be two points $t_1, t_2 \in T$ that form a central angle $\angle(t_1, p^*, t_2)$ in $[\frac{2}{3}\pi, \pi]$. Thus, for any $\varphi \in [0, \frac{2}{3}\pi]$, we can perform a single fold along a line through p^* that maps t_2 to t_2' such that the central angle $\angle(t_1, p^*, t_2')$ is φ.

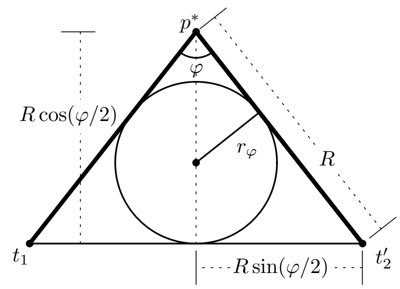

Figure 2: Parameters for calculating the 1-fold cover factor for convex S.

Now, after folding, consider a cover of the three points t_1, p^*, t_2' by cS for some $c > 0$. As each member of cS is convex, in covering the triangle $\Delta(t_1, p^*, t_2')$, it also covers the largest circle C_Δ contained in $\Delta(t_1, p^*, t_2')$; let r_φ be the radius of this circle, see Figure 2. Using elementary geometry we obtain $r_\varphi = \frac{R}{2}\frac{\sin(\varphi)}{1+\sin(\varphi/2)}$, which is maximized at $\varphi = 2\arctan\big(((\sqrt{5}-1)/2)^{1/2}\big) \approx 76.345°$, giving $r_\varphi = \kappa R$ as the radius of C_Δ. Because the largest circle covered by cS has radius cr, and C_Δ is covered by cS, we conclude that $c \ge \kappa R/r$. □

2.2 Cover factors for specific polygons

In this section we determine $c_1^*(S)$ when S is an equilateral triangle or a square. These two cases illustrate analysis techniques that could in theory be extended to other polygons, except that the number of cases explodes, especially for non-convex shapes.

An important subproblem is to fix the folded shape F and compute, for the given shape S, the smallest c such that cS covers F. With four degrees of freedom for translation, rotation, and scaling, we expect that four first-order contacts between the boundaries of S and F will define the minimum c. In polygons, these will be four pairs consisting of a vertex of F and an edge of S that it lies on. A enclosing triangle, therefore, has some edge touching two vertices of F [5, Lemma 2]; a enclosing square either has some edge touching two vertices or each edge touching one [4].

In the full paper we say more about how these structural characterizations support the use of rotating calipers to compute minimum enclosing shapes. (E.g., an appealing direct construction of the square through four points, which is unique when it exists, is the solution to problem 20 in Kovanova and Radul's list of "Jewish problems" [6]: for points A, B, C, D in ccw order, construct BD' perpendicular and of equal length to AC; If $D' \neq D$, then two sides of the square must be parallel to DD'.) In what follows we show that the folds that define $c_1^*(S)$ (that maximize the minimum scale factor) are characterized by having multiple equal-sized enclosing shapes.

2.2.1 Equilateral triangle

The example that establishes the maximum 1-fold cover factor of an equilateral triangle is nicely symmetric.

Theorem 4 *The* 1*-fold cover factor of an equilateral triangle,* $c_1^*(\triangle)$*, is* $4/3$*.*

Proof. Let S be the triangle of side length 2 with vertices $(\pm 1, 0)$ and $(0, \sqrt{3})$. We begin by showing that any single fold can be covered by scaling to at most $4/3$.

By symmetry, we may assume that we fold along a line $y = mx + b$ that intersects both edges incident on $(0, \sqrt{3})$; let P be the image of this vertex in the folded state $S' \in \mathcal{F}_1(S)$. Consider three cases for the location of the image P and the resulting minimum enclosing equilateral triangle, depicted in Figure 3.

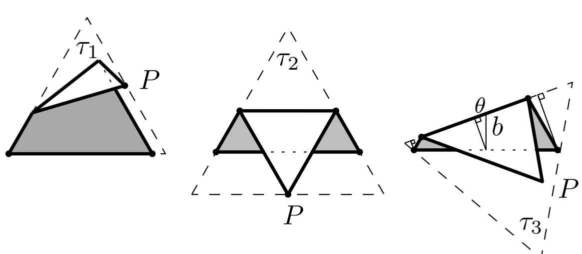

Figure 3: Cases for enclosing triangle depending on P. Point $P \in \tau_3$ should be below $P \in \tau_2$, but then small triangles mentioned in the proof are even harder to see.

First, suppose that P is on or above the x-axis. By symmetry, we may assume that P lies in the wedge formed by extending both edges of S incident on vertex $(-1, 0)$ to rays from $(-1, 0)$. Because P has distance at most 2 from $(-1, 0)$, scaling S about $(-1, 0)$ by $2/\sqrt{3} < 4/3$ creates an enclosing equilateral triangle τ_1.

Second, suppose that the image $P = (p_x, p_y)$ has $-\sqrt{3}/3 \leq p_y \leq 0$. Consider the enclosing triangle τ_2 obtained by scaling S about $(0, \sqrt{3})$ until the horizontal edge touches P. The scale factor for this triangle is $\frac{\sqrt{3}-p_y}{\sqrt{3}} = 1 - p_y/\sqrt{3} \leq 4/3$.

Finally, suppose that $P = (p_x, p_y)$ has $p_y \leq -\sqrt{3}/3$. From the previous case, the scale factor for enclosing triangle τ_2 is $1 - p_y/\sqrt{3} \geq 4/3$. So instead consider an enclosing triangle τ_3 with an edge e along the fold line, which we can parameterize by its y-intercept $b \leq \sqrt{3}/3$ and angle from horizontal θ. Draw perpendiculars to e through vertices $(\pm 1, 0)$ to form two small 30-60-90 triangles. Edge e is composed of the short sides of these triangles plus the projection of the base edge of S, so e has length $(2 + 2b/\sqrt{3}) \cos\theta$. Thus, the scale factor of triangle τ_2 is $(1 + b/\sqrt{3}) \cos\theta \leq 4/3 \cos\theta \leq 4/3$.

These cases show that $c_1^*(\triangle) \leq 4/3$, and also reveal necessary conditions for equality: the fold line angle $\theta = 0$ and intercept $b = \sqrt{3}/3$, so $P = (0, -\sqrt{3}/3)$. To show that these are sufficient, we must check one more candidate for enclosing triangle.

Consider τ_4, with edge incident to $P = (0, -\sqrt{3}/3)$ and $(-1, 0)$. The length of this edge is the sum of sides of two 30-60-90 triangles, marked α and β in Figure 4. The scale factor $(\alpha + \beta)/2 = \sqrt{3}/9 + 2\sqrt{3}/3 = 7\sqrt{3}/9 > 4/3$. Thus, τ_4 is *not* a minimum enclosing triangle, and $c_1^*(\triangle) = 4/3$, as determined by τ_2 and τ_3.

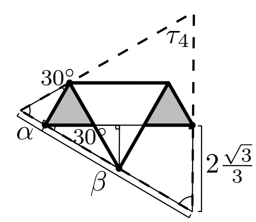

Figure 4: Not a min enclosing triangle.

This completes the proof. $\square$

2.2.2 Square

For squares, the optimal fold is astonishingly complex, and is neither symmetric, nor rational. For the unit square $[0, 1]^2$, the vertex $(0, 1)$ folds to a location whose y coordinate is the root of a degree twelve polynomial: $\Phi(x) = 40x^{12} + 508x^{11} + 1071x^{10} + 930x^9 - 265x^8 - 1464x^7 - 1450x^6 - 524x^5 + 58x^4 + 76x^3 + 3x^2 - 6x - 1$. This polynomial will arise because the optimal fold has *three* distinct minimum enclosing squares. Let ρ denote the largest (and only positive) real root of $\Phi(x)$, which is approximately 1.105224.

Let $S = \{(x, y) : 0 \leq x, y \leq 1\}$ denote the axis-parallel unit square and consider some $F \in \mathcal{F}_1(S)$ such that $F \neq S$. Note that F is a simple polygon that is uniquely determined (up to symmetry) by a fold line ℓ.

Proposition 5 *The polygon F can be covered by S, unless fold line ℓ intersects S in the relative interior of two opposite sides.*

Proof. If ℓ does not intersect the interior of S then $F \cong S$. Otherwise ℓ intersects ∂S in exactly two points. If these points lie on adjacent sides of S, then folding along ℓ reflects the triangle formed by these sides and ℓ inside the portion of the square S on the opposite side of ℓ. Therefore, F can be covered by S. $\square$

We are interested in a fold line ℓ that maximizes the smallest enclosing square of F. Using symmetry with Proposition 5, we can assume:

(1) the line ℓ intersects both horizontal sides of S (else rotate by 90°);
(2) the slope of ℓ is negative (else reflect vertically);
(3) ℓ intersects the top side of S left of the midpoint $(1/2, 1)$ (else rotate by 180°).

If we imagine F as the result of folding the part of S to the left of ℓ over to the right, then we can parameterize ℓ by the image $P = (p_x, p_y)$ of the top left corner $(0, 1)$ of S under this fold. Under the above assumptions, a line ℓ that passes (almost) through $(1/2, 1)$ and $(1, 0)$ would maximize p_y. Therefore $0 < p_x < 4/5$ and so $1 < p_y < \sqrt{2p_x - {p_x}^2 + 1} < 7/5$.

Denote the two points of intersection between ℓ and ∂S by $B = (b_x, 0)$ and $T = (t_x, 1)$ and denote the image of the bottom-left corner $(0, 0)$ of S under the fold across ℓ by $Q = (q_x, q_y)$. If $q_x > 1$, then the convex hull $\mathcal{CH}(F)$ of F is the hexagon $B, (1, 0), Q, (1, 1), P, T$, else Q does not appear on $\partial(\mathcal{CH}(F))$ and it is only a pentagon. Note that in any case the width of F in the y-direction is greater than one, whereas the width in the x-direction is less than one.

For a given $P = (p_x, p_y)$, we have

$$\ell : y = -\frac{p_x}{p_y - 1}x + \frac{{p_x}^2 + {p_y}^2 - 1}{2(p_y - 1)},$$

$$T = \Big(\frac{{p_x}^2 + (p_y - 1)^2}{2p_x}, 1\Big),$$

$$B = \Big(\frac{{p_x}^2 + {p_y}^2 - 1}{2p_x}, 0\Big), \text{ and}$$

$$Q = \Big(\frac{p_x({p_x}^2 + {p_y}^2 - 1)}{{p_x}^2 + (p_y - 1)^2}, \frac{({p_x}^2 + {p_y}^2 - 1)(p_y - 1)}{{p_x}^2 + (p_y - 1)^2}\Big).$$

What does a smallest enclosing square σ of F look like? For the upper bound on the cover factor we consider three enclosing squares (Figure 5).

σ_1 is the smallest axis-parallel enclosing square, which has points B and $(1, 0)$ on the bottom side, P on the top, T on the left, and no point on the right.

σ_2 has points P and $(1, 1)$ on one side, B on the opposite side, and T on a third side.

σ_3 has points B, $(1, 0)$, $(1, 1)$, and T appearing in this order, each on a different side of σ_3.

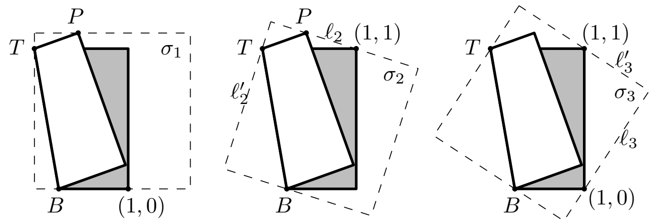

Figure 5: Three minimum enclosing squares for F.

Theorem 6 *The 1-fold cover factor of a square, $c_1^*(\square)$, is ρ, the real root of the degree twelve polynomial Φ.*

Proof. The effort goes into showing that, for each folded shape F, one of the three enclosing squares σ_i, $i \in \{1, 2, 3\}$, as defined above, has side length at most ρ.

Denote the side length of a square σ by $|\sigma|$. For a start it is easy to see that $|\sigma_1| = p_y < 7/5$, which provides a first upper bound.

For σ_2 we have to consider the distance $\mathrm{d}(B, \ell_2)$, where ℓ_2 is the line through P and $(1, 1)$ and the distances $\mathrm{d}((1, 0), \ell_2')$ and $\mathrm{d}(Q, \ell_2')$, where ℓ_2' is the line orthogonal to ℓ_2 through T. Noting that

$$\mathrm{d}(B, \ell_2) = \frac{\left|{p_x}^2 p_y + {p_y}^3 + {p_x}^2 - 2p_x p_y - {p_y}^2 - p_y + 1\right|}{2p_x\sqrt{(p_x - 1)^2 + (p_y - 1)^2}}$$

$$\mathrm{d}((1,0), \ell_2') = \frac{\left|{p_y}^2 p_x + {p_x}^3 - {p_y}^2 - 3{p_x}^2 + 2p_y + p_x - 1\right|}{2p_x\sqrt{(p_x - 1)^2 + (p_y - 1)^2}},$$

it can be checked that the former dominates the latter for $p_y \le \frac{1}{2}(1 + \sqrt{4p_x - 4{p_x}^2 + 1})$ and that $\mathrm{d}((1, 0), \ell_2') > p_y$ for $\frac{1}{2}(1 + \sqrt{4p_x - 4{p_x}^2 + 1}) < p_y < \sqrt{2p_x - {p_x}^2 + 1}$ (and so $|\sigma_1| \le |\sigma_2|$ in such a case). Exactly the same holds if $\mathrm{d}((1, 0), \ell_2')$ is replaced by

$$\mathrm{d}(Q, \ell_2') = \frac{|N_1|}{2p_x(1 + (p_x - p_y)^2)\sqrt{(p_x - 1)^2 + (p_y - 1)^2}},$$

where $N_1 = {p_x}^5 + 2{p_x}^3{p_y}^2 + p_x{p_y}^4 - {p_x}^4 - 2{p_x}^3 p_y - 2p_x{p_y}^3 + {p_y}^4 - 4{p_x}^2 p_y - 4{p_y}^3 + 4{p_x}^2 + 2p_x p_y + 6{p_y}^2 - p_x - 4p_y + 1$. This verifies that σ_2 is enclosing, with side length $|\sigma_2| = \mathrm{d}(B, \ell_2)$.

For σ_3 we consider a line $\ell_3 : y = m(x - 1)$ through $(1, 0)$, for some $m > 0$ and the orthogonal line $\ell_3' : y = (m + 1 - x)/m$ through $(1, 1)$. If σ_3 is a smallest enclosing square, then $\mathrm{d}(T, \ell_3) = \mathrm{d}(B, \ell_3')$. For our range of parameters, the only solution is

$$m = \frac{{p_x}^2 + {p_y}^2 - 1}{{p_x}^2 + (p_y - 1)^2},$$

which yields

$$|\sigma_3| = \mathrm{d}(T, \ell_3) = \frac{\sqrt{2}|N_2|}{4p_x\sqrt{D_2}},$$

where $N_2 = p_x{}^4 + 2p_x{}^2p_y{}^2 + p_y{}^4 - 4p_x{}^3 - 2p_x{}^2p_y - 4p_xp_y{}^2 - 2p_y{}^3 + 4p_xp_y + 2p_y - 1$ and $D_2 = p_x{}^4 + 2p_x{}^2p_y{}^2 + p_y{}^4 - 2p_x{}^2p_y - 2p_y{}^3 + 2p_y{}^2 - 2p_y + 1$.

Because we choose the smallest square among σ_1, σ_2, and σ_3, the claim certainly holds for $|\sigma_1| = p_y \leq \rho$.

It can be checked that $|\sigma_2| \leq \rho$, for all P with $\rho < p_y < \sqrt{2p_x - p_x{}^2 + 1}$, except for a small region $\mathcal{R}$. This region $\mathcal{R}$ is bounded from below by the line $y = \rho$ and from above by the curve $\gamma : |\sigma_2| = \rho$ (the branch of this curve that lies in $\{(x, y) : \rho \leq y < \frac{1}{2}(1 + \sqrt{4x - 4x^2 + 1}\})$. The curve γ intersects the line $y = \rho$ at two points, whose x-coordinates are approximately 0.67969 and 0.77126, respectively. The more interesting of these two is the first point of intersection, which can be described exactly as the smallest positive real root x_ρ of the polynomial $40x^{12} - 116x^{11} - 1045x^{10} + 4756x^9 - 10,244x^8 + 7260x^7 - 8392x^6 - 184x^5 + 620x^4 - 160x^3 + 1088x^2 - 192x + 256$. For the fold defined by $P = (x_\rho, \rho)$ we have $|\sigma_1| = |\sigma_2| = |\sigma_3| = \rho$, while for all other points in $\mathcal{R}$ the corresponding value for $|\sigma_3|$ is strictly less than ρ.

It can also be checked that $|\sigma_3| < \rho$, for any P with $p_y > \rho$ and $\frac{1}{2}(1 + \sqrt{4p_x - 4p_x{}^2 + 1}) < p_y < \sqrt{2p_x - p_x{}^2 + 1}$ (above we committed to using σ_2 only if $p_y \leq \frac{1}{2}(1 + \sqrt{4p_x - 4p_x{}^2 + 1})$).

Altogether it follows that $\min\{|\sigma_i| : i \in \{1, 2, 3\}\} \leq \rho \approx 1.105224446$, as claimed.

Using rotating calipers, one can verify that all other enclosing squares are larger, giving the equality. □

2.2.3 Polygons and single folds

In the full paper we prove that any polygon (a finite cyclic sequence of vertices and edges with no self-intersections) can be made larger with a single fold. The following lemma is in contrast to observations in Section 3.2 for disks and a family of shapes related to disks.

Lemma 7 *For every plane polygon P, the 1-fold cover factor, $c_1^*(P)$, is greater than* 1.

The idea of the proof is to look for finite sets of structures in P that, if not destroyed by folding, can be covered only by members of that set. For example, the set of diameters in a polygon is finite because the maximum distance D is realized by pairs of vertices, and any diametral pair still at distance D in the folded state F must be covered by a diameter of P, possibly itself.

For a quick example, consider the class of polygons P in which there exist vertices that participate in two or more diametral pairs. (E.g., for odd n, every vertex of a regular n-gon.) Choose as our structure two diametral pairs, pq and qr, that minimize $\theta = \angle pqr$. Fold along a line trisecting θ, reflecting qr to create qr' in the folded shape F. This modified structure has angle $\angle pqr' = \theta/3$ between two diameters; by minimality of θ, it cannot be covered by P.

The proof repeatedly identifies classes of polygons by structures found in the neighborhoods of diameters, until every polygon is in some class. Modifications to these structures show that $c_1^*(P) > 1$ for all polygons.

3 Arbitrary Folds

3.1 Simply connected shapes

In this section we show that, for a simply connected shape S, there is a lower bound for the origami cover factor $c^*(S)$ that is within a constant factor of the upper bound given by Lemma 2.

Theorem 8 *Let S be a simply connected shape with inradius r, geodesic radius R, and geodesic diameter D. Then $\kappa R/r \leq D/(2\pi r) \leq c^*(S) \leq R/r$ for $\kappa = \sqrt{3}/(2\pi) \approx 0.27566$.*

Proof. Again, the upper bound is from Lemma 2. The basic idea for the lower bound is to find a path in S that can be folded into a large circle, which must then be covered by a scaled copy of the incircle of S. Here, for brevity, we use a path of length D, the geodesic diameter.

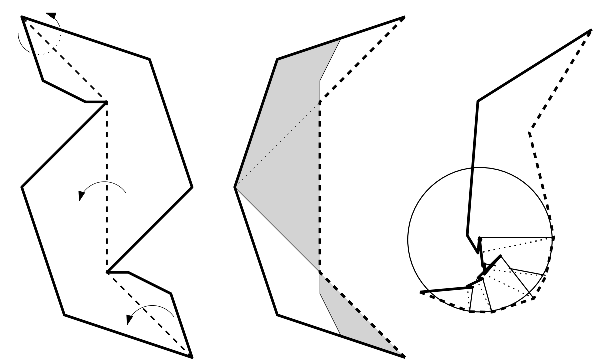

Figure 6: For Theorem 8, folding inflection edges to make a generalized spiral, then crimping to approximate a circle that must be covered by the incircle.

A *generalized spiral* is a simply connected region composed of consistently orientable plane patches having a distinguished shortest path γ that follows the boundary and never turns to the left. A generalized spiral may overlap itself if projected onto a plane, but we can think of it as embedded in a covering space of the plane.

Ordinarily, a diameter path γ will alternate between sequences of left turns and right turns at boundary points; a portion of the path between opposite turns is a line segment that we can call an *inflection edge*. We can simply fold along every inflection edge, gluing doubled layers along these edges, to turn γ into a path that goes only straight or to the right. Folding any non-boundary edges creates a generalized spiral with path γ. These folds are along lines of the geodesic path, so γ remains a shortest path between its endpoints.

We fold the generalized spiral into a left-turning circle with circumference approaching the length of γ. If we sweep a paired point and normal vector along γ, we can think of painting a portion of the generalized spiral with fibers that each start on γ and grow orthogonal to a local tangent (because γ is a shortest path) and that are disjoint (because the sweep in position and angle is monotonic). We construct a circle whose circumference is arbitrarily close to the length of γ by crimp folds that align successive fibers of γ with the circle center. Figure 6 shows an example. It does not matter how far the fibers extend towards or beyond the circle; in order to cover the boundary of the circle, the inradius r must be scaled to the circle radius, which is $D/(2\pi)$. $\square$

3.2 Disks with bumps

Because the radius of a disk is simultaneously the inradius and the geodesic radius, Lemma 2 implies that the cover factor of a disk, $c^*(\bigcirc)$, is 1. It is interesting to note that there are other shapes S with $c^*(S) = 1$; here is one simple family.

In a unit disk centered at C with a chord AB, choose a point D between C and the midpoint of AB. Add the disk centered at D of radius $|AD|$. Thus, we have a family of shapes $S_{d,e}$, parameterized by two distances, $d = |CD|$ and $e =$ distance from C to chord AB, satisfying $0 < d \le e < 1$. See Figure 7.

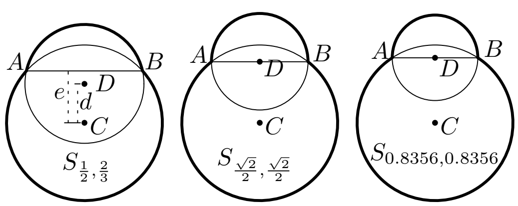

Figure 7: Shapes $S_{d,e}$ with $c^*(S_{d,e}) = 1$.

Lemma 9 *The shape $S_{d,e}$, with $0 < d \le e < 1$, has origami cover factor $c^*(S_{d,e}) = 1$.*

Proof. Shape $S_{d,e}$ is the union of a unit disk centered at C and a disk centered at D whose radius we denote r. Note that by construction the boundaries of the disks intersect at A and B. This shape also covers all disks of radius r that are centered between C and D.

Now, in an arbitrary folded state $S'_{d,e}$, consider the locations of these centers, C' and D'. Placing a unit disk centered at C' and a radius r disk centered at D' will cover all points of $S'_{d,e}$. Because $|C'D'| \le |CD|$, this pair of disks will be covered by placing a copy of $S_{d,e}$ with C at C' and D on the ray $\overrightarrow{C'D'}$. $\square$

Choose any $d \in (0,1)$ and for all $e \in [d,1)$ shape $S_{d,d}$ covers $S_{d,e}$, so these extremal members of the family have AB as the diameter of the smaller disk. Just for the sake of curiosity, the example with $d = e = \sqrt{2}/2$ minimizes the ratio of inradius to circumradius, $R/r = (1 + \sin\theta + \cos\theta)/2 \approx 0.8284$, and the example with $d \approx 0.8356$ minimizes the fraction of the circumcircle covered, $(\pi(1 + \sin^2\theta) + \sin 2\theta - \theta)/(\pi R^2) \approx 0.7819$.

4 Open Problems

The most interesting questions are whether $c^*(\triangle) = c_1^*(\triangle)$ and $c^*(\square) = c_1^*(\square)$, and whether we can completely characterize those shapes with origami or 1-fold cover factor of unity.

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