Interlocked Open and Closed Linkages with Few Joints

Erik D. Demaine
MIT Laboratory for Computer Science,
200 Technology Square, Cambridge, MA 02139, USA.

Stefan Langerman*
School of Computer Science, McGill University, 3480 University Street, Suite 318,
Montreal, QC, H3A 2A7, Canada

Joseph O’Rourke 1
Department of Computer Science,
Smith College, Northampton, MA 01063, USA.

Jack Snoeyink 2
Department of Computer Science,
UNC Chapel Hill, Chapel Hill, NC, 27599-3175, USA.

Abstract

We study collections of linkages in 3-space that are interlocked in the sense that the linkages cannot be separated without one bar crossing through another. We explore pairs of linkages, one open chain and one closed chain, each with a small number of joints, and determine which can be interlocked. In particular, we show that a triangle and an open 4-chain can interlock, a quadrilateral and an open 3-chain can interlock, but a triangle and an open 3-chain cannot interlock.

Key words: Linkages in $S^3$, Knots, Configurations
2000 MSC: 57M25, 70B15, 68U05

* Corresponding author.
Email addresses: edemaine@mit.edu (Erik D. Demaine), sl@cgm.cs.mcgill.ca (Stefan Langerman), orourke@cs.smith.edu (Joseph O’Rourke), snoeyink@cs.unc.edu (Jack Snoeyink).

1 Supported by NSF Distinguished Teaching Scholars award DUE-0123154.
2 Partially supported by NSF grants 9988742 and 0076984.
1 Introduction

Consider a simple polygonal chain, either an open arc or a closed polygon, that is embedded in 3-space. We view the vertices of the chain (except the endpoints of an open chain) as universal joints, and the edges of the chain as rigid bars. We call a chain with \( k \) bars a \( k \)-chain. A motion of the chain is a motion of the vertices that preserves the length of the bars, and never causes bars to cross. In particular, a straightening of an open chain is a motion that makes all joint angles become 180°. We say that a collection of disjoint, simple chains can be separated if, for any distance \( d \), there is a motion whose result is that every pair of points on different chains has distance at least \( d \). If a collection cannot be separated, we say that its chains are interlocked. If a single chain cannot be straightened, we say that it is locked.

It is known that a single, open chain in 3-space, having as few as 5 bars, can be locked [1,2]. Other classes of chains are known to be unlocked, but the complexity of deciding whether a given chain can be unlocked is not known. One decision procedure applies the roadmap algorithm for general motion planning [3,4], which runs in exponential time.

Our work is inspired by a question posed by Anna Lubiw [5]: Into how many pieces must a chain be cut so that the pieces can be separated and straightened? This problem is motivated by protein molecules, which can be modeled by polygonal chains, and, according to some theories, temporarily split apart in order to reach the minimum-energy folding.

We can observe easy upper and lower bounds for Lubiw’s problem: some \( n \)-chains require cutting at least \( \lfloor (n-1)/4 \rfloor \) vertices for separation, and no chain requires cutting of more than \( \lfloor (n-1)/2 \rfloor \) vertices. The lower bound is obtained by concatenating many copies of the 5-bar “knitting needles” example from [1,2], each sharing one bar with the next as in Fig. 1. Observe that each copy of the locked 5-bar chain must have one of its four interior vertices cut. The upper bound is obtained by cutting every second joint of a chain, and observing

![Fig. 1. An \( n = 17 \) bar chain that requires cutting at least \( \lfloor (n-1)/4 \rfloor = 4 \) vertices to separate.](image)

that the resulting 2-bar pieces (“hairpins”) can be rigidly separated arbitrarily far by dilating from a point, because the pieces are starshaped sets. This separation motion dates back at least to de Bruijn in 1954 [6], where he used it to prove separability of convex objects; the same motion was shown to apply to the more general situation of starshaped objects by Dawson in 1984 [7], and the algorithmic side of this result is described by Toussaint in 1985 [8]. See also [9].
While Lubiw’s problem motivated our original interest in interlocked open chains, we explore here interlocking for combinations of open and closed chains. In the next section, we resolve how many bars are needed by each chain in order to obtain an interlocked pair, as summarized in Table 1.

<table>
<thead>
<tr>
<th>Sec</th>
<th>Chain 1</th>
<th>Chain 2</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>closed triangle</td>
<td>open 3-chain</td>
<td>Cannot Interlock</td>
</tr>
<tr>
<td>3.1</td>
<td>closed triangle</td>
<td>open 4-chain</td>
<td>Can Interlock</td>
</tr>
<tr>
<td>3.2</td>
<td>closed quadrilateral</td>
<td>open 3-chain</td>
<td>Can Interlock</td>
</tr>
</tbody>
</table>

Table 1

Our results on when an open chain and a closed chain can interlock. A claim that a $k$-chain can interlock holds also for any $l$-chain with $l > k$, and a claim that a $k$-chain cannot interlock holds also for any $l$-chain with $l < k$.

2 Triangle and 3-chain Cannot Interlock

We begin by showing that a triangle and a 3-chain cannot interlock. As we will see later, this is in some sense a maximal non-interlocking configuration.

**Theorem 1** An open 3-chain cannot interlock with a triangle.

**Proof.** We follow this notation: $\triangle abc$ lies in plane $H$, and the 3-chain $C$ has vertices $(p_0, p_1, p_2, p_3)$ and bars $(l_0, l_1, l_2)$. First assume $C$ is not planar; otherwise, make $C$ non-planar by a small motion. Let $L_i$ be the support line of $l_i$ and define points $q_i = L_i \cap H$.

1. Bar $l_1$ intersects the closed $\triangle abc$. In this case, it is possible to move bar $l_0$ and bar $l_2$ within the plane that it forms with $l_1$ so that the angle at the joint shared with $l_1$ is arbitrarily close to either 0 or $\pi$, because one of the two wedges spanned by these two motions does not intersect any other edge. Once both end bars have been moved to that position, $C$ is arbitrarily close to a single bar which can be translated in the direction $p_1p_2$.

2. Bar $l_1$ does not intersect the closed $\triangle abc$. Because configuration $C$ is non-self-intersecting, we can assume that the points $\{q_0, p_1, p_2, q_2\}$ do not lie on a common plane, or equivalently $\{q_0, q_1, q_2\}$ are not collinear. Denote the line containing $q_0$ and $q_2$ by $Q_{02}$, as in Fig. 2. In fact, for any position of $l_1$ such that $(L_1 \cap H) \not\subset Q_{02}$, the lines containing $q_0p_1$ and $p_2q_2$ do not intersect, and do not intersect the edges of $\triangle abc$. Thus the motion that translates $l_1$ in a direction orthogonal to $Q_{02}$ and parallel to $H$, away from $\triangle abc$, while maintaining $L_0$ and $L_2$ through the original points $q_0$ and $q_2$, will avoid self-intersection.³

³ See http://www.cs.smith.edu/~orourke/Interlocked/ for an animation of this motion.
3 Interlocked Examples and the Topological Method

Our two proofs that chains are interlocked follow a similar structure in what we call the topological method. We imagine tying the two ends of the open chain with a long rope near infinity, which defines a topological link (multicomponent knot) [10, p. 17]. For the two chains to separate, they must form the trivial link (referred to as $0^2_1$; see later). First we show that before this happens, the ends of the open chain must get close to the closed chain. Second we argue that this proximity is impossible before changing the topology of the link. Finally we prove that this circularity leads to a contradiction, so the chains are interlocked.

![Diagram of interlocked chains](image)

Fig. 3. The first few two-component links.

To make connections to known mathematics for links, we will refer to some links by their numbers from standard tables. See [10, p. 287] or [11, p. 1086]. Tables of links are often
organized by (minimum) crossing number. The superscript in the link notation is the number of components, for us always 2. The subscript is an arbitrary table index. See Fig. 3.

3.1 Triangle and 4-chain

We begin with the configuration illustrated in Fig. 4.

**Theorem 2** A triangle can interlock with a 4-chain.

**Proof.** We choose the following notation for the configuration of Fig. 4: A triangle $abc$ lies in a plane $H$, with $H^+$ the halfspace above and $H^-$ the halfspace below $H$. Let the circumcircle of $\triangle abc$ have center $o$, and radius $r$.

The 4-chain alternates points and bars $p_0, l_0, p_1, l_1, \ldots, l_3, p_4$ with the following placements: $p_0$ is in $H^-$, bar $l_0$ crosses the interior of $\triangle abc$, and ends at a point $p_1$ above $o$. Bar $l_1$ crosses the interior of $\triangle abc$ again, so $p_2 \in H^-$. Bar $l_2$ crosses $H$ outside of $\triangle abc$, and $l_3$ crosses the wedge formed by $l_0$ and $l_1$ above $H$. So $\{p_0, p_2\} \subset H^-$ and $\{p_1, p_3, p_4\} \subset H^+$.

![Fig. 4. A triangle and a 4-chain can lock.](image)

Let $R$ be the real number $r + |l_1| + |l_2|$, and set the length of $l_0$ and $l_3$ to $20R$. Consider the open ball $B$ of radius $15R$, and the ball $B'$ of radius $4R$, both centered at $o$. Initially, $p_0$ and $p_4$ lie outside of $B$, while $a, b, c, p_1, p_2$ and $p_3$ all lie inside $B' \subset B$. As long as $p_0$ and $p_4$ stay outside $B$ and all other vertices stay inside $B$, we can attach a sufficiently long unknotted string between $p_0$ and $p_4$ that remains outside $B$, and thus is never crossed by any of the bars, and our configuration is equivalent to the link $5_2^2$. The non-interlocked configuration corresponds to two separable unknotted $0^2_1$, so any motion separating this configuration would require $p_0$ or $p_4$ to enter the ball $B$ or $p_1, p_2$, or $p_3$ to leave $B$.

Consider the first event when any $p_i, i = 0, \ldots, 4$ touches the boundary of $B$. Then before or at that event, points $p_1, p_2$ and $p_3$ must be out of $B'$ but still inside $B$: When $p_0$ touches

---

4. Link images produced by Robert Scharein's knotplot program

http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html
$B$, point $p_1$ must be exterior to $B'$ by at least $R$, and therefore $p_2$ and $p_3$ are also exterior to $B'$. See Fig. 5. The same applies for when $p_4$ touches the boundary of $B$. When any one of $p_1$, $p_2$ or $p_3$ touches the boundary, the other two are at least at a distance $14R$ from $o$ and so are outside of $B'$. Since we consider the first such event, there must be an instant before that when all three points are outside $B'$ but still inside $B$.

![Diagram](image)

Fig. 5. When $p_0$ touches $B$, point $p_1$, $p_2$ and $p_3$ must be exterior to $B'$.

At this time, the only elements possibly inside $B'$, besides $\triangle abc$, are the two bars $l_0$ and $l_3$. Then either one of $l_0$ and $l_3$ crosses the interior of $\triangle abc$, or both do, or neither do. The first case corresponds to a link $2_1^2$ and the third case to two separable unknots $0_1^2$; neither of these are equivalent to our starting configuration (in the knot theoretical sense). Since the rope and the bars have not crossed, the topology of the configuration cannot have changed and so these cases lead to a contradiction.

The case in which both $l_0$ and $l_3$ cross $\triangle abc$ requires a careful analysis. Because end vertices $p_0$ and $p_1$ are still outside of the open ball $B$, we can replace the string joining them by a great arc $\gamma$ on the boundary of $B$. Let $T$ be the plane parallel to $l_0$ and $l_3$, and passing through $o$. Consider the orthogonal projection of the 4-bar linkage onto $T$. Note that in the projection, the lengths of bars $l_0$ and $l_3$ are preserved, and all other segment lengths are at most their original lengths. Let $q_0$ be the intersection of $l_0$ and plane $H$. The triangle $\triangle abc$ is contained in a ball of radius $2R$ centered at $q_0$, and joints $p_1$, $p_2$ and $p_3$ lie in a ball of radius $R$ centered at $p_1$. Since $p_1$ is outside $B'$ and $q_0$ is inside the circumcircle of $\triangle abc$, the distance between those two points is larger than $3R$, and that distance is preserved in the projection. Thus, the projections of the two balls are disjoint and we can separate the projections of $p_1$, $p_2$ and $p_3$ from the projections of $p_0$, $p_1$ and $\triangle abc$ by a line (This separation is necessary to exclude cases such as the one shown in Fig. 6.), and the two bars $l_1$ and $l_2$ can be replaced by a single bar joining $p_1$ and $p_3$ without changing the topology of the link. By enumerating all possible above/below combinations for the crossings in that projection, we can infer that configuration is equivalent to $0_1^2$, which is two separated, unknotted links, or to $4_1^2$, which is shown in Fig. 7. But neither of these are topologically equivalent to our starting configuration, so this first event could never happen.
Fig. 6. This configuration is incompatible with the fact that $p_0$ or $p_4$ touches the boundary of $B$.

Fig. 7. The link $A_1^2$, formed when bars $l_0$ and $l_3$ both pass through the interior of $\triangle abc$. (Not to scale; gray segments indicate omissions.) Joints $\{p_1, p_2, p_3\}$ can be separated from $\{a, b, c, p_0, p_4\}$.

Note that a similar argument can be used to show that the chains in Fig. 6 are interlocked as well.

3.2 Quadrilateral and 3-chain

In the following, we will use what is known as the linking number of a two component link. We first arbitrarily orient both components of the link. Then each crossing drawn in the projection of the link has one of two types, associated with a value $+1$ or $-1$. See Fig. 8.

Fig. 8. Sign of a crossing.
The linking number of the link is half the sum of the values of all crossings between the different components; crossings of a component with itself are not counted. For example, the link $3_1^2$ has 5 crossings, but only four of them involve both components. The sum of the values of the four crossings is 0, which yields a linking number of 0. Note that if the orientation of one of the components is reversed, then the linking number is negated. It can be proved using some elementary knot theory that the linking number of an oriented link is an invariant, that is, it has the same value for all drawings of the oriented link [10, p. 21].

**Theorem 3** A 4-gon can interlock with a 3-chain.

**PROOF.** Let the 4-gon be $abcd$, and again use $(l_0, l_1, l_2)$ and $(p_0, p_1, p_2, p_3)$ to represent the bars and vertices of the 3-chain. Starting with the configuration of Fig. 9, let $R = |ab| + |bc| + |cd| + |l_1|$ and set the length of $l_0$ and $l_2$ to $20R$. Consider the open ball $B$ of radius $15R$, the ball $B'$ of radius $4R$, and the ball $B''$ of radius $R$, all three centered at $a$. As in the previous proof, we connect $p_0$ to $p_3$ by a string exterior to $B$. The resulting link is now $6_2^2$. We again argue that in order to separate the 4-gon from the 3-chain, $p_0$ or $p_3$ has to enter the ball $B$ or $p_1$ or $p_2$ have to leave $B$. Before that, there must be an instant when $p_0$ and $p_3$ are still outside $B$, $p_1$ and $p_2$ are still inside $B$ but out of $B'$, and the only elements possibly inside $B'$, besides $abcd$, are the two edges $l_0$ and $l_2$.

![Fig. 9. A quadrilateral and a 3-chain can lock.](image)

If neither $l_0$ nor $l_2$ intersects $B''$, then the configuration is the link $0_1^2$, contradicting that the topology cannot have changed. If one of the two end bars, say $l_0$, intersects $B''$, let $q_0$ be a point of $l_0 \cap B''$. We project the configuration onto a plane parallel to $l_0$ and $l_2$, preserving the distances along those two bars. As in the previous proof, because the length of the segment $q_0p_1$ is preserved in the projection, only the interiors of $l_0$ and $l_2$ can intersect the projection of $B''$. This implies that the linking number of the configuration will be the sum of the values induced by $l_0$ and $abcd$, and the values induced by $l_2$ and $abcd$, divided by 2. Notice that the total of the values induced by a straight edge and a 4-gon is at most 2, and so the linking number of the configuration is at most $(2 + 2)/2 = 2$. But the linking number of $6_2^2$ is 3. Because the linking number is an invariant, the topology of the configuration must have changed, a contradiction.
4 Open Problems

Many open problems remain in the context of interlocking pairs of open chains, which have close connections to the motivating problem of Lubiw. For each value of $i$, what is the smallest $j$ for which an $i$-chain can interlock with a $j$-chain?

The topological method of Theorems 2 and 3, where we used a “rope” to close one open chain to form a topological linkage, does not easily extend to pairs of open chains. Two ropes would be needed, and their potential interactions would need to be controlled. To extend this work, therefore, we will be investigating a geometric method that establishes a collection of geometric facts and shows that there can be no first violation. We believe that we can use such a method to establish three conjectures: that a 3-chain can interlock with a 4-chain, that three 3-chains can interlock, but that two 3-chains cannot interlock even in the presence of any finite number of 2-chains.

The proof of Theorem 3 depends upon a tetrahedron formed by the 4-gon, and does not show that a 3-chain and a $k$-gon can interlock for any $k > 4$. In fact, adding any small edge to the 4-gon would allow the 3-chain to escape. On the other hand, our conjecture that a 3-chain can interlock with a 4-chain, once established, would imply that a 3-chain can interlock with a $k$-gon for any $k \geq 5$ by connecting the endpoints of the 4-chain with one or more edges.

Chains that model physical objects, such as robot arms or protein backbones, often have restrictions placed on the motion of a joint. There are a number of interesting problems for open and closed chains under various restrictions on motions. For example, we conjecture that a rigid, open 3-chain can interlock with a flexible, open 3-chain.

Acknowledgements

This work was initiated in the Waterloo algorithmic open-problem session, and continued at CCCG 2000, with contributions by Therese Biedl, Hamish Carr, Eowyn Čenek, Timothy Chan, Beenish Chaudry, Martin Demaine, Rudolf Fleischer, John Iacono, Anna Lubiw, Dessislava Michaylova, Veronica Morales, Katherine Sinclair, Geetika Tewari, and Ming-wei Wang. The authors wish to thank the anonymous referees for many useful comments and for suggesting a more elegant proof for Theorem 1.

References


