

Simple Folding is Strongly NP-Complete

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Deciding whether a crease pattern can fold into a non-self-intersecting flat-folded state is known to be strongly NP-complete [BH96], but if the folding must be formed using a sequence of simple folds with prescribed mountain/valley assignment, the decision problem has only been shown weakly NP-complete [ABD⁺04] via reduction from Partition. We close this gap by proving the latter problem strongly NP-complete, even for orthogonal polygons with only orthogonal valley creases.

Theorem 1. *The problem of deciding simple foldability of an orthogonal piece of paper with an orthogonal mountain-valley pattern is strongly NP-complete.*

The reduction is from 3-Partition. Given an instance of 3-Partition with integers $A = \{a_1, \dots, a_n\}$ to be partitioned into $n/3$ triples each with sum $\sum A/(n/3) = t$, construct an orthogonal polygon with orthogonal valley creases as shown in Figure 1. We assume each a_i is close to $t/3$ and divisible by $2n$: if not, add a large number and multiply by $2n$ so that they are.

There are four main functional sections of the polygon. On the left is the Bar, a $2 : 2\infty$ rectangle of paper without creases that is very long ($\infty = 10nt$). The Staircase encodes the a_i s in order as a series of steps with height equal to their value plus one. Each step i contains two creases c_{2i-1}, c_{2i} that when both folded will raise the Bar by exactly $2a_i$. The Wrapper section is a horizontal rectangle of length $2n/3$ with vertical valley creases d_i (d_1 being the right most crease) dividing the Wrapper into unit squares. The Cage on the right bounds a polygonal area.

The construction forces the Bar to wrap inside the Cage $n/3$ times, each time shifted up by distance $2t$ (note that ∞ is chosen large to ensure that the Staircase never intersects the Cage polygon while wrapping). To prove the claim, we first prove the Wrapper must fold its vertical creases in order from right to left. If this were not the case, then there exists some first crease d_i to be folded whose right neighbor d_{i-1} has not yet been folded. But d_i has at least two squares of unfolded paper to its left that will cover d_{i-1} when folded, making d_{i-1} impossible to fold using simple folds without violating the mountain/valley assignment, contradicting our model. Because the Wrapper executes its folds from right to left, the Bar must pass through the Cage $n/3$ times sequentially from the rightmost slot to the leftmost, with each subsequent slot shifted up by $2t$.

If the 3-Partition instance has a positive solution, then the polygon has a simple folding: just fold the creases associated with the a_i s in one of the satisfying triples, then fold the Bar through the Cage along Wrapper creases, and repeat. Further, if the polygon has a simple folding, the 3-Partition instance has a positive solution because the Staircase must be folded on both creases from exactly three a_i s between each wrap. To see this, all a_i s are close to $t/3$ so exactly three a_i sections must be flipped from their original orientation to shift by $2t$. Further, because a_i s are divisible by $2n$, no unit section between a_i s can flip if the total height raises each time by t , which is also divisible by $2n$. So the a_i s chosen at each step correspond to triplets of the 3-Partition instance that sum to t .

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The reduction is polynomial because the entire constructed polygon is bounded by a $3\infty \times 4n$ rectangle. Lastly, the problem is in NP because given a certificate of the crease folding order, each fold can be simulated and checked in polynomial time. \square

Because this reduction only ever folds through one layer at a time, the reduction works in the one-layer, some-layers, and all-layers models of simple folds. Our reduction extends to folding a square polygon with creases constrained to angles at multiples of 45° using the same construction as described in [ABD⁺04]. We also adapt this reduction to crease patterns without mountain/valley assignment under the some-layers and all-layers simple fold model. Determining NP-hardness, weak or strong, remains open for unassigned mountain/valley assignment under the one-layer simple folds model.

References

- [ABD⁺04] Esther M. Arkin, Michael A. Bender, Erik D. Demaine, Martin L. Demaine, Joseph S. B. Mitchell, Saurabh Sethia, and Steven S. Skiena. When can you fold a map? *Computational Geometry: Theory and Applications*, 29(1):23–46, September 2004.
- [BH96] Marshall Bern and Barry Hayes. The complexity of flat origami. In *Proc. 7th ACM-SIAM Symp. Discrete Algorithms*, ACM/SIAM, pages 175–183, New York/Philadelphia, 1996.

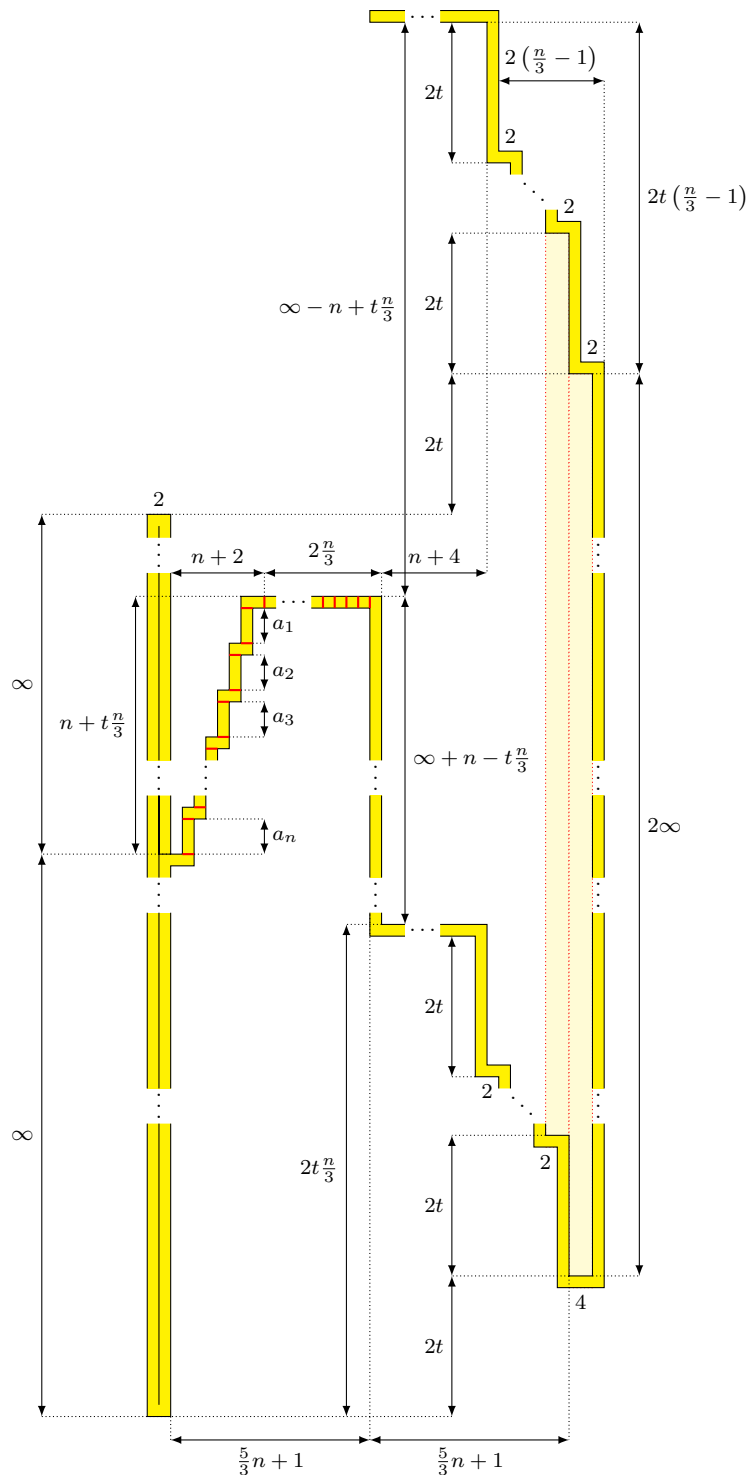


Figure 1: An orthogonal simple polygon with orthogonally aligned valley creases (in red) constructed from an instance of 3-Partition that can be folded using simple folds if and only if the instance of 3-Partition is valid.