Minimal Unfoldable Polyhedron

Hugo A. Akitaya, Erik D. Demaine, David Eppstein, Tomohiro Tachi, and Ryuhei Uehara

1 Department of Computer Science, Tufts University, USA.
2 CSAIL, MIT, USA.
3 Department of Computer Science, University of California, Irvine, USA.
4 Department of General Systems Studies, The University of Tokyo, Japan.
5 School of Information Science, JAIST, Japan.

1 Introduction

In computational origami, one of the most remarkable problems asks if every convex polyhedron has an edge-unfolding to a simple and nonoverlapping polygon [3, Open Problem 21.1]. Here, an edge-unfolding is a set of edges of the polyhedron such that cutting along these edges unfolds the polyhedron. There are several evidences for/against the conjecture (see [3, Sections 22.2 and 22.3]); however, it is far from to be settled. As an evidence for this conjecture, it is known that any convex polyhedron \( P \) is edge-unfoldable if \( P \) is a prismoid or a dome; however, they form a quite limited set of convex polyhedra. On the other hand, as an evidence against this conjecture, there are several edge-unfoldable non-convex polyhedra. In this context, a key property of a polyhedron is topological convexity; a polyhedron \( P \) is topologically convex iff there is a convex polyhedron \( P' \) such that the graph induced by its vertices and edges is isomorphic to the one induced by \( P \). In fact, if we do not restrict to topologically convex polyhedra, it is easy to make an edge-unfoldable polyhedron with 7 faces: take a tetrahedron and add a tetrahedral bump in the middle of one of its faces.

Research on small edge-unfoldable nonconvex polyhedra gives us insight of the open problem. In [5], Grünbaum gives a polyhedron with 13 vertices and 13 faces that is not edge-unfoldable. We sharpen such edge-unfoldability of nonconvex polyhedra:

**Theorem 1.1** (1) There exists a topologically convex polyhedron with 7 vertices and 6 faces that is not edge-unfoldable. (2) There exists a topologically convex polyhedron with 6 vertices and 7 faces that is not edge-unfoldable. (3) Any polyhedron with less than 6 vertices or 6 faces is edge-unfoldable.

That is, we give much smaller edge-unfoldable polyhedra, and we also prove that they are optimal with respect to the number of faces and vertices, respectively. Note that they are not only topologically convex, but also all their faces are simple polygons (or simply connected and no holes).

2 Edge-unfoldable polyhedra

In order to prove Theorem 1.1(1), we show a polyhedron \( P \) with 7 vertices and 6 faces s.t. any edge-unfolding of \( P \) causes an overlapping. It is as shown in Figure 1: \( P \) has an apex \( G \), and the base of \( P \) consists of one triangle \( ABF \) and one concave pentagon \( BCDEF \). The side of \( P \) consists of one triangle \( CDG \) and \( EDG \) and 2 concave quadrilaterals \( ABCG \) and \( AFEG \). We make \( P \) symmetric to its mirror image across the plane \( AGD \). The key properties of the polyhedron \( P \) are the following: (a) total angle around \( D > 360^\circ \), (b) total angle around \( G > 360^\circ \), (c) \( \angle CBF + \angle CBA > 360^\circ \), and (d) \( \angle ABF + \angle CBA > 300^\circ \) and \( BF < BC \).

Let \( \mathcal{G} = (V, E) \) be the graph induced by \( P \). Then any edge-unfolding of \( P \) induces a spanning tree \( T \) of \( \mathcal{G} \). Let \( T \) be any spanning tree of \( \mathcal{G} \). We show that any edge-unfolding given by \( T \) produces overlapping. Since \( T \) is a tree, it has at least 2 leaves. However, from the properties (a)-(c), none of \( B, F, D, G \) can be a leaf since overlapping occurs at leaves. On the other hand, vertices \( A, C, E \) cannot be three leaves since we cannot have any spanning tree \( T \) that contains the edge \( BF \) with 3 leaves \( A, C, E \). Thus \( T \) has only 2 leaves, i.e., \( T \) is a Hamilton path of \( \mathcal{G} \). In most cases, \( ABCG \) overlaps with \( CBFED \) at vertex \( B \) by (c). The other case is shown in Figure 2. Thus the polyhedron \( P \) in Figure 1 has no edge-unfolding without overlapping. In order to prove Theorem 1.1(2), we show a polyhedron \( P \) with 6 ver-
3 Edge-unfoldable polyhedra with less than 6 vertices or 6 faces

In order to prove Theorem 1.1(3), we first recall that DiBiase established that all convex polyhedra with 4, 5, or 6 vertices can be edge-unfolded [4]. Based on this result, there are only two cases to be considered; \( P \) is (a) a nonconvex square pyramid (with 5 faces and 5 vertices), or (b) a nonconvex polyhedron that consists of 6 triangles and 5 vertices with degree sequence 3,3,4,4,4. The proof that these two polyhedra are edge-unfoldable is omitted here.

4 Concluding remarks

We investigated minimal edge-unfoldable polyhedra. We show two edge-unfoldable polyhedra; one has 6 faces and 7 vertices, and the other has 7 faces and 6 vertices. They are optimal with respect to the number of faces and vertices for topologically convex polyhedra. Grünbaum shows an edge-unfoldable convex faced polyhedron with 13 faces and 13 vertices [5], and Bern et al. give an edge-ununfoldable triangular faced polyhedron with 36 faces and 20 vertices [1]. Improving these upper bounds and/or showing lower bounds are future work.

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References