A Disk-Packing Algorithm for an Origami Magic Trick

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Abstract

We present an algorithm for a magic trick. Given a polygon with holes P, our algorithm determines a folding of a rectangular sheet of paper such that a single straight cut suffices to cut out P. This paper is a simplification and improvement of a paper first published in *Fun with Algorithms* [10].

1 Introduction

The great Harry Houdini was one of the first to perform the following magic trick: fold a sheet of paper so that a single straight cut produces a cut-out of a rabbit, a dog, or whatever else one likes. Whereas Houdini only published a method for a five-pointed star [13] (a method probably known to Betsy Ross [12]), Martin Gardner [11] posed the question of cutting out more complex shapes. Demaine and Demaine [5] stated this question more formally: given a polygon with holes P (possibly with more than one connected component) and a rectangle R large enough to contain P, find a "flat folding" of R such that the cross-section of the folding with a perpendicular plane is the boundary of P. More intuitively, a single straight cut of the flat folding produces something that unfolds to P. A flat folding [4, 14] is a mathematical notion, abstracting folded paper to a nonstretchable, non-self-penetrating, zero-thickness, piecewise-linear surface in $\mathbb{R}^3$.

Demaine et al. [6, 7] have proposed a solution to this *cut-out problem*, based on propagating paths of folds out to the boundary of the rectangle R. Here we give a more "local" solution, based on disk packing. Our strategy is to pack disks on R so that disk centers induce a mixed triangulation/quadrangulation respecting the boundary of polygon P. We fold each triangle or quadrilateral interior (exterior) to P upwards (respectively, downwards) from the plane of the paper, taking care that neighboring polygons agree on crease orientations. A cut through the plane of the paper now separates interior from exterior.

Disk packing has previously been used to compute triangulations [1] and quadrangulations [3] with special properties. Disk packing, or more precisely disk placement, has also been applied to origami design, most notably by Lang [15]. In fact, the result in this paper

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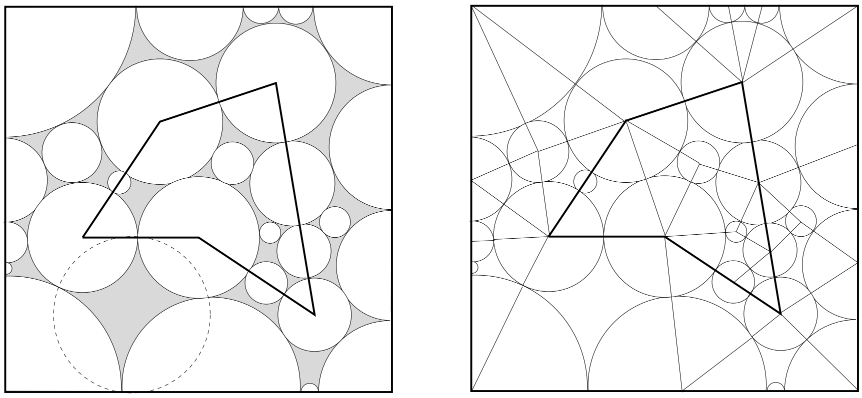

Figure 1: (a) A disk packing respecting the boundary of the polygon. Vertices of 4-gaps are cocircular. (b) Induced triangles and quadrilaterals.

is in some sense a fusion of a quadrangulation algorithm from Bern and Eppstein [3] with an origami design algorithm from Lang [15].

2 Disk Packing

Let P be a polygon with holes, strictly contained in a rectangle R. We think of P as boundary along with interior. Let PR denote the planar straight line graph that is the union of the boundary of P and the boundary of R. In this section, we sketch how to pack disks such that each edge of PR is a union of radii of disks, and such that the disks induce a partition of R into triangles and quadrilaterals. Our solution is closely related to some mesh generation algorithms [1, 3].

The disk packing starts with interior-disjoint disks. We call a connected portion of R minus the disks a *gap*. We call a gap bounded by three arcs a *3-gap* and one bounded by four arcs a *4-gap*. We begin by centering a disk at each vertex, including the corners of R. At vertex v, we place a disk of radius one-half the distance from v to the nearest edge of PR not incident to v. We introduce a subdivision vertex (a degree-2 vertex with a straight angle) at each intersection of a disk boundary and an edge of PR.

Now consider the edges of (the modified) PR that are not covered by disks. Call such an edge *crowded* if its diameter disk intersects the diameter disk of another edge of PR. We mark each crowded edge, and then split each crowded edge by adding its midpoint. We continue marking and splitting in any order until no edges of PR are crowded. We then add the diameter disk of each PR edge so that each edge is a union of diameters of disks as required. Strictly speaking, only the edges of P need be covered by disks, but we include the boundary of R for the sake of neatness.

Next we add disks until all gaps between disks are either 3-gaps or 4-gaps. This can be done by computing the Voronoi diagram of the disks placed so far, and repeatedly placing a maximal-radius disk at a Voronoi vertex and then updating the Voronoi diagram. Bern et al. [1] give an $O(n \log^2 n)$ algorithm and Eppstein [9] an $O(n \log n)$ algorithm, where n denotes the number of disks.

Figure 1(a) gives an example disk packing, not precisely the same as the one that would be computed by the algorithm just sketched. By adding edges between the centers of tangent disks, the disk packing induces a decomposition of R into triangles and quadrilaterals as shown in Figure 1(b).

3 Molecules

A *molecule* is a (typically flat) folding of a polygon that can be used as a building block in larger origamis. We shall fold the triangles in the decomposition of R with *rabbit ear molecules*. In the rabbit ear molecule, a mountain fold meets each of the triangle's vertices; these folds lie along the angle bisectors of the triangle so that the boundary of the triangle is coplanar in the folded "starfish". A valley fold meets each of the triangle's sides at the points of tangency of the disks; these all fold to a vertical *spine*, perpendicular to the original plane of the paper. The meeting point of the six folds, which becomes the tip of the spine in the folded configuration, is the in-center of the original triangle.

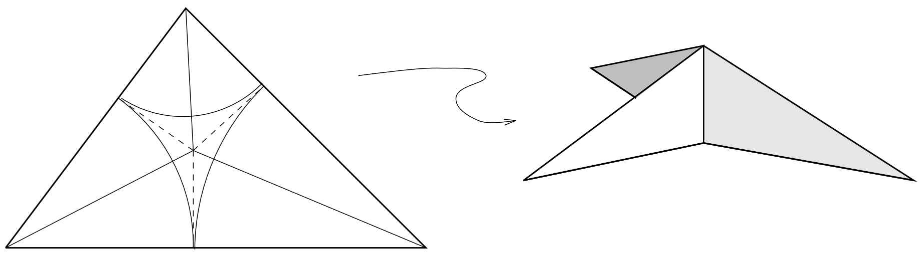

Figure 2: A rabbit ear molecule folds into a three-armed "starfish".

At this point, we regard the orientations of the valley folds as changeable: in the larger origami some of them may be reversed from their initial assignment. For example, to form a flat origami from a single rabbit ear, one could reverse one of the valleys into a mountain in order to satisfy Maekawa's theorem.[1] The arms of the starfish all point the same direction away from the spine in the flat origami, and the boundary of the original triangle is collinear.

We shall fold the quadrilaterals as shown in Figure 3. This folding is an improvement, suggested by Robert Lang, of our original method of folding quadrilaterals [10]. In this *gusset molecule* [15], mountain folds extend some distance along the angle bisectors to a *gusset*, a quadrilateral inside the original quadrilateral, shown shaded in Figure 3. The gusset is triangulated with one of its two diagonals, a valley fold, and each of the halves of the overall quadrilateral is folded in a sort of rabbit ear molecule. This folding of the quadrilaterals enjoys the same property as the folding of the triangles: the valley folds from points of tangency all meet at a central spine, perpendicular to the plane of the paper. Again we regard the orientations of these folds as changeable. In the larger origami, we may reverse one of the valleys in order to form a flat folding with all arms pointing in the same direction. Notice that such a reversal also sends a crease (a mountain-valley two-edge path, shown dotted in Figure 3) across the central gusset.

[1]Maekawa's theorem for flat origami [4, 14] states that at any vertex interior to the paper the number of mountains minus the number of valleys must be plus or minus two.

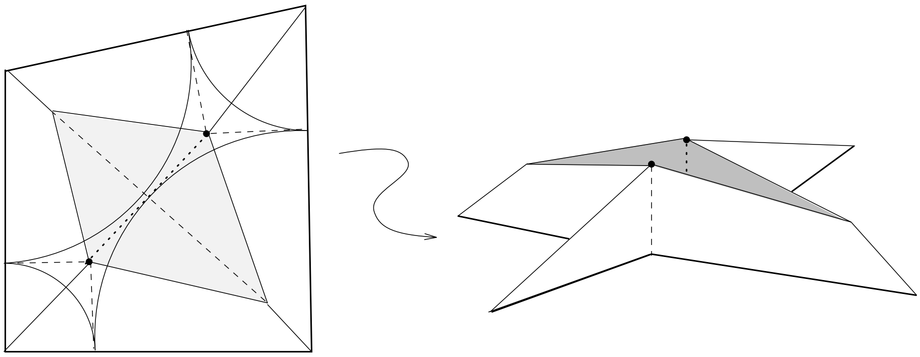

Figure 3: We fold a quadrilateral into a four-armed starfish with a central valley.

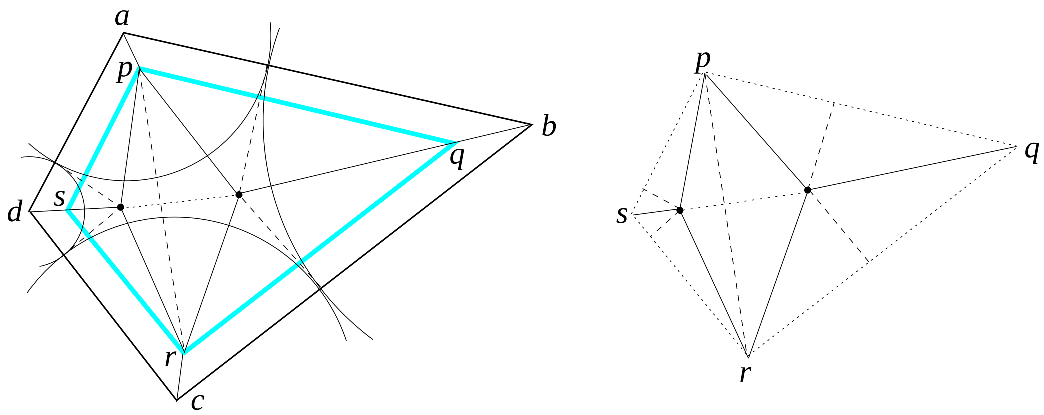

Figure 4: (a) The two unconstrained vertices of the gusset may be chosen to lie on an inset quadrilateral. (b) The inset quadrilateral is folded with two rabbit-ear molecules.

Two of the vertices of the gusset, shown by dots in Figure 3, are fixed by the requirement that valley folds extend perpendicularly from the points of tangency. We refer to these vertices as the *perpendicular points*. The other two vertices of the gusset are not completely constrained. They must, however, lie on the angle bisectors of the quadrilateral in order for the boundary of the quadrilateral to fold to a common plane.

A nice way [15] to locate the the unconstrained vertices—p and r in Figure 4(a)—is to place them at the vertices of an *inset quadrilateral*, a quadrilateral inside the overall quadrilateral, with sides parallel and equidistant to the sides of the original quadrilateral. In Figure 4(a) the original quadrilateral is $abcd$ and the inset quadrilateral is $pqrs$. When the gusset molecule is folded, the inset quadrilateral will form a small starfish whose central valley exactly reaches "sea level", that is, $pqrs$ and pr fold to the same plane. In fact, the gusset folding restricted to $pqrs$ is just two rabbit-ear molecules, as shown in Figure 4(b). Hence, the perpendicular points must lie at the in-centers of triangles pqr and prs, and this requirement determines the size of $pqrs$.

We now argue that all quadrilaterals induced by 4-gaps—all the quadrilaterals that we use—can be folded with the gusset molecule. What we must show is that the triangles

pqr and prs with in-centers at the perpendicular points do indeed lie within $abcd$, in other words, that the requirements of the gusset molecule are not in conflict with each other.

First assume that the perpendicular points are distinct, and consider the line L through the perpendicular points. Line L is the line of equal power distance[2] from the disks centered at a and c, and hence passes between these disks. The bisector of the angle between L and the valley fold perpendicular to bc fixes the r. Since L passes above the disk at c, r lies above c along the angle bisector at c. Thus $pqrs$ does indeed lie within $abcd$. In the extreme case that the disks at a and c touch each other, $pqrs$ equals $abcd$ and the gusset molecule reduces to two rabbit-ear molecules.

What if the perpendicular points coincide? For this extreme case, we use a special property [1] of 4-gaps: the points of tangency of four disks, tangent in a cycle, are cocircular. Figure 1(a) shows the circle for one 4-gap. This property implies that the angle bisectors of the quadrilateral all meet at a common point o, namely the center of the circle through the tangencies. So in the extreme case that the perpendicular points coincide, $pqrs$ shrinks to point o, and the valleys from the points of tangency and the mountains along the angle bisectors all meet at one flat-foldable point.

4 Joining Molecules

We now show how to assign final orientations to creases, so that neighboring molecules fit together and each vertex satisfies Maekawa's theorem. This fills in (a special case of) a missing step in Lang's algorithm [15].

We are aiming for a final folding of R that resembles a book of flaps, something like the rightmost picture in Figure 6. More precisely, the folding will look like two books of triangular flaps, one above and one below the original plane of the paper. The molecules (triangles and quadrilaterals) inside P will form the top book, whereas those outside P will form the bottom book. The boundary of P itself will not be folded, and the polygons crossing the boundary, each containing a triangle from two different original molecules, will thus belong to both books.

Angle bisector edges inside P will be mountains and those outside P will be valleys. Other edges of the crease pattern receive *default orientations*, subject to reversal in a final matching step. The default orientation of a *tangency edge* (an edge to a point of tangency) or a *side edge* (an edge along the side of a triangle or quadrilateral) is valley inside P and mountain outside P. Side edges lying along the boundary of P are not folded at all.

At this point, each vertex of the crease pattern has an equal number of mountains and valleys. The vertices interior to R inside P need one more mountain, whereas those outside P need one more valley, in order that molecules fold to their assigned half-spaces, above or below the original plane of the paper. (Vertices on the boundary of P can have an excess of either mountains or valleys.) Let G be the planar graph obtained from the decomposition by removing all angle bisector edges and all edges along the boundary of P. We would like to find a set of edges M—a matching—such that each vertex of G lying in the interior of R is incident to exactly one edge of M. By reversing the orientations of the edges of M,

[2]The power distance [1] from a point to a circle is the square of the usual distance minus the radius of the circle squared. For points outside the circle it is the same as the tangential distance to the circle squared.

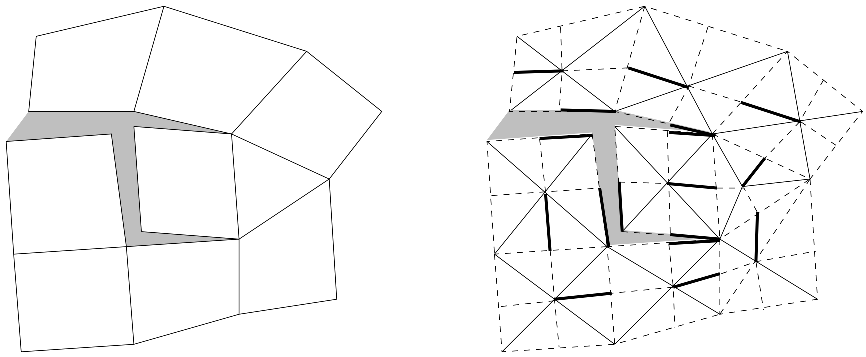

Figure 5: (a) Cutting out a tree T_C (shaded) spanning interior corners leaves a tree of molecules T_M. Roots are at the upper left. (b) The matching consists of side edges from corners to parents in T_C and tangency edges from molecule centers to parents in T_M. Assignments shown assume all molecules are inside P.

we ensure that each vertex satisfies Maekawa's theorem. All vertices, even the ones along P, which lost two edges each from the original decomposition of R, also satisfy Kawasaki's theorem.[3]

We now show how to solve the matching problem using dual spanning trees. Let T_C be a tree of side edges such that: T_C includes no edges along the boundary of R or P; T_C spans all interior corners of molecules; and T_C spans exactly one corner along the boundary of R, which we consider to be its root. If we were to cut the paper along T_C, we would obtain a tree of molecules T_M, as shown in Figure 5(a). We root T_M at one of the molecules incident to the root of T_C. The matching M contains two types of edges: each tangency edge from the center of a molecule to the side of its parent in T_M (along with one such edge inside the root molecule), and each side edge from a corner to its parent (a tangency point) in tree T_C. See Figure 5(b).

To picture the effect of this choice of M on the eventual flat folding, imagine that we have actually cut along the edges of T_C. Imagine building up the flat folding molecule by molecule in a preorder traversal of T_M. The root molecule of T_M folds to a book of flaps with collinear edges lying along the original plane of the paper. Each child molecule adds a "pamphlet" of three or four flaps between two flaps of the book we have constructed so far. The cover and back cover of the pamphlet are glued to their adjacent pages, so that a quadrilateral thickens two old flaps and adds two new flaps.

We continue gluing pamphlets between flaps of the growing book as we go down the tree. Whenever we cross the boundary of P, we glue the next pamphlet above or below—rather than between flaps of—its parent molecule, so that the boundary of P is not itself folded. When we are done joining all the molecules we indeed have two books of flaps, one above and one below the original plane of the paper.

Now imagine taping the cut edges back together in a postorder traversal of T_C. Before taping, the cut leading to a leaf of T_C, say inside P, defines the bottom edge of two adjacent "armpits", as shown in Figure 6. (An armpit consists of one layer from each of two adjacent

[3]Kawasaki's theorem for flat origami [4, 14] states that at any vertex interior to the paper the sum of alternate angles must be 180°.

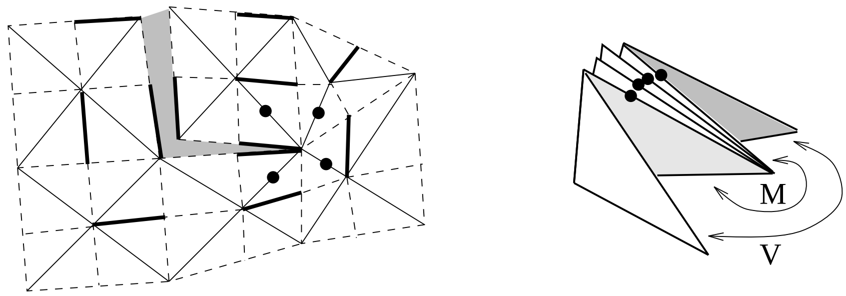

Figure 6: Taping together a cut leading to a leaf of T_C amounts to joining two "armpits" in the book of flaps.

flaps.) Taping together the first and last layer of the intervening flap forms a mountain fold, agreeing with the orientation we gave to side edges in the matching. Taping together the remaining two sides of the cut forms a valley fold, agreeing with the default orientation of side edges inside P. Taping a cut leading to a leaf of T_C closes two armpits and reduces the number of flaps in the book by two. We can continue taping cuts all the way up T_C. Since each taping joins armpits adjacent at the time of the taping, there can be no "crossed" pair of tapings, or put another way, no place where the paper is forced to penetrate itself. Altogether the taping completes a crease pattern on paper R that can be folded flat so that P lies above, and its complement $R \setminus P$ lies below, the original plane of the paper.

5 Fattening the Polygon

At this point, we have a degenerate solution to the cut-out problem. A cut through the original plane of the paper separates P from its complement. Unfortunately, it also cuts P into its constituent molecules. A cut very slightly below the original plane of the paper leaves P intact, while adding a small "rim" to P.

We can remove the degeneracy by fattening the boundary of P into a narrow "ribbon" as shown in Figure 7. The boundaries of the ribbon are slightly inside and outside the original P; vertices of P are moved in or out along angle bisectors. (We actually saw this ribbon construction already: a gusset molecule is two adjacent rabbit ear molecules surrounded by a ribbon!) The width of the ribbon must be smaller than the *minimum feature size* of the polygon, the minimum distance between a vertex of P and an edge not incident to that vertex.

We modify the disk packing step so that it packs partial disks (sectors) around the boundary of ribbon, such that interior and exterior sectors match up. Creases between corresponding subdivision points cross the ribbon at right angles, whereas creases between corresponding vertices cross at angle bisectors, so that each vertex still satisfies Kawasaki's theorem. Notice that interior and exterior sectors centered on corresponding vertices have slightly different radii.

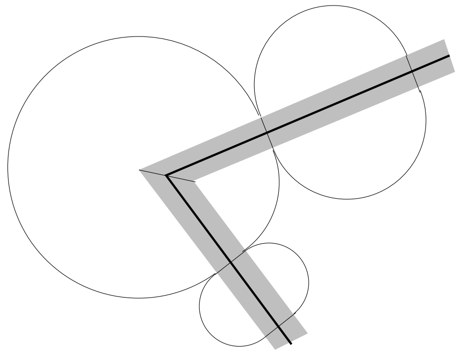

Figure 7: Fattening the polygon into a ribbon lets P survive the cut intact.

6 Discussion

We have given an algorithm for the cut-out problem. More precisely, we have given an algorithm for computing a crease pattern with a flat folding that solves the cut-out problem. We have not described how to actually transform the crease pattern into the flat folding.

Is our algorithm usable? The answer is a qualified yes. Figure 8 gives a crease pattern for a fish cut-out that is not too hard to fold. In this crease pattern, we have taken a number of shortcuts to make the algorithm more practical. First, we have used only three-sided and special four-sided gaps, ones in which perpendiculars from the center vertex o happen to meet the sides at the points of tangency. Second, we have not packed the disks all the way to the boundary of the paper, only far enough that radiating folds do not meet within the page. Third, we have not fattened the polygon, and hence the cut should be placed slightly below original plane of the paper, so that the interior remains connected.

The number of creases used by our algorithm is not really excessive, linear in the number of disks in the initial disk packing. The number of disks, in turn, depends upon a fairly natural complexity measure of the polygon. Define the *local feature size* $LFS(p)$ at a point p on an edge e of P to be the distance to the closest edge that is not adjacent to e [16]. The local feature size is small at narrow necks of the polygon. It is not hard to see that the number of disks around the boundary of P is $O(\int_{\partial P} 1/|LFS(p)|)$, where the integral is over the boundary of P. The number of additional disks needed to fill out the square is linear in the disks around the boundary of P, because each new disk reduces the number of sides of the gap into which it is placed.

The algorithm of this paper can be generalized to the problem in which the input is a planar straight-line graph G, and a single cut must cut along all the edges of G. An interesting open question asks whether there is a polynomial-size solution (polynomial in the number of original vertices of P or G) for the cut-out problem. A solution using disk packing may shed some light on two other computational geometry problems: simultaneous inside-outside nonobtuse triangulation [2] and conforming Delaunay triangulation [8].

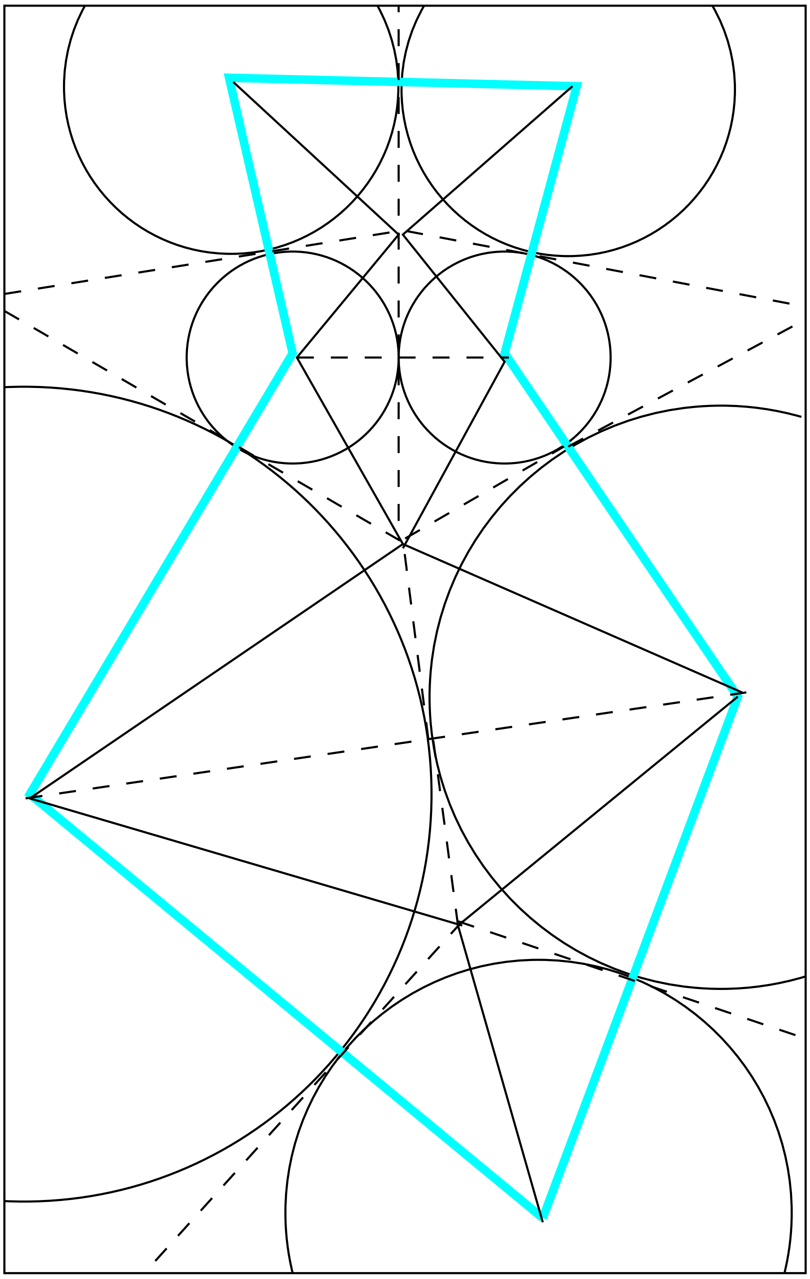

Figure 8: An example for the reader to try. This “mounted marlin” design incorporates some practical shortcuts. For example, packing the exterior with disks is unnecessary, because radiating folds do not collide.

Acknowledgments

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