Path Puzzles: Discrete Tomography with a Path Constraint is Hard

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Path Puzzles are a type of logic puzzle introduced in Roderick Kimball's 2013 book [5]. A puzzle consists of a (rectangular) grid of cells with two exits (or "doors") on the boundary and numerical constraints on some subset of the rows and columns. A solution consists of a single non-intersecting path which starts and ends at two boundary doors and which passes through a number of cells in each constrained row and column equal to the given numerical clue. Figure 1 shows some example path puzzles and Figure 4 shows their (unique) solutions. Many variations of path puzzles are given in [5] and elsewhere, for example using non-rectangular grids, grid-internal constraints, and additional candidate doors, but these generalizations make the problem only harder.

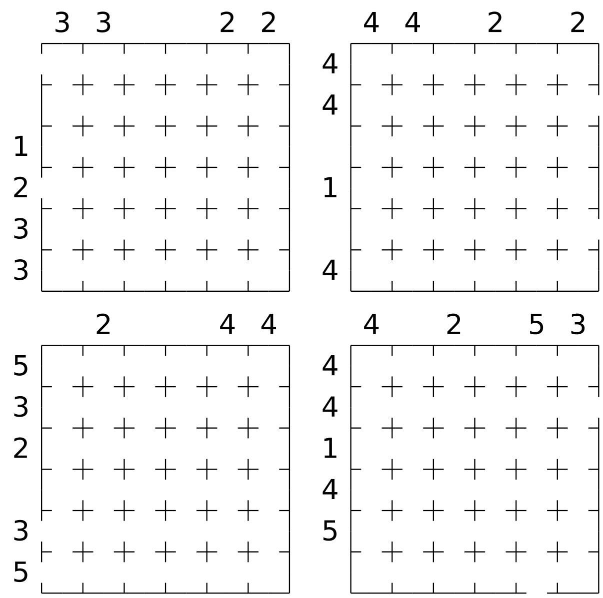

Figure 1: Four path puzzles. Solutions in Figure 4 on the next page.

A path puzzle can be seen as 2-dimensional *discrete tomography* [3] problem with partial information (not all row and column sums) and an additional Hamiltonicity constraint on the output image. Vanilla 2-dimensional discrete tomography is known to have efficient (polynomial-time) algorithms [3], though it becomes hard under certain connectivity constraints on the output image [2].

Our results. Unlike 2-dimensional discrete tomography, we show that path puzzles (with partial information and the added Hamiltonicity constraint) are in fact NP-complete. In fact, we prove the stronger results that path puzzles are Another Solution Problem (ASP) hard and (to count solutions) #P-complete. Figure 2 shows the chain of reductions we prove. To preserve hardness for the ASP and #P classes, our reductions are *parsimonious*, that is, they preserve the number of solutions between the source and target problem instances, generally by showing a one-to-one correspondence thereof. We start from the source problem of Positive Exactly-1-In-3-SAT which is known to be ASP-hard [6] and (to count solutions) #P-complete [4]. We newly establish ASP-hardness and #P-completeness for 3-Dimensional Matching, Numerical 4-Dimensional Matching, Numerical 3-Dimensional Matching, and a new problem Length Offsets, in addition to Path Puz-

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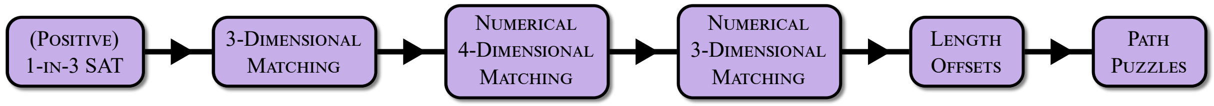

Figure 2: The chain of reductions used in our proof.

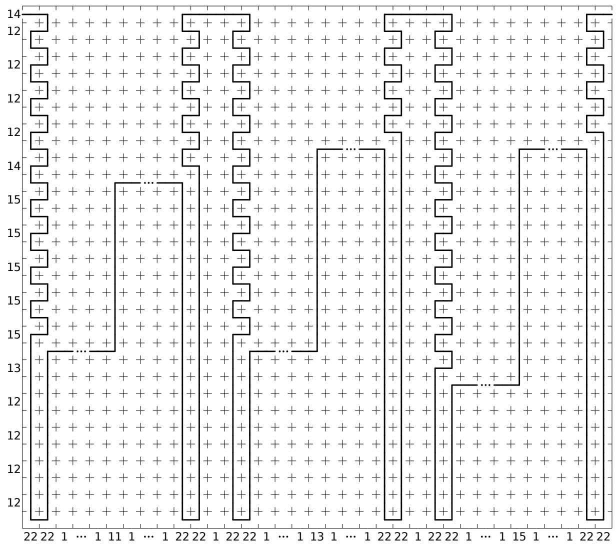

Figure 3: The reduction Numerical 3-Dimensional Matching → Length Offsets → Path Puzzles (with intended solution) representing Numerical 3-Dimensional Matching instance $X = \{5, 6, 7\}$, $Y = \{4, 5, 5\}$, $Z = \{4, 4, 5\}$, and target sum $t = 15$. Ellipses elide sections of $6n = 18$ columns each labeled 1.

zles. Figure 3 gives a flavor of our reductions.

We also present a path puzzles font—a set of 26 path puzzles whose (unique) solutions depict the alphabet. Figures 1 and 4 show the J, C, D, and G puzzles.

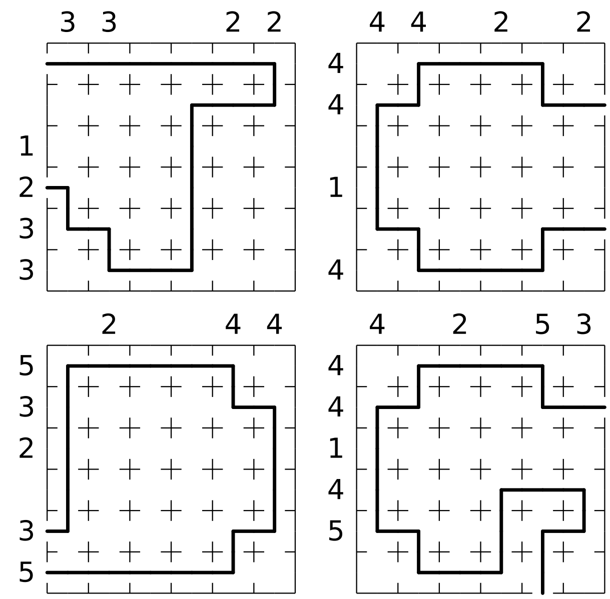

Figure 4: Solutions to the path puzzles in Figure 1. What can you spell?

References

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