Polyhedral Characterization of Reversible Hinged Dissections

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Abstract. We prove that two polygons A and B have a reversible hinged dissection (a chain hinged dissection that reverses inside and outside boundaries when folding between A and B) if and only if A and B are two non-crossing nets of a common polyhedron. Furthermore, *monotone* hinged dissections (where all hinges rotate in the same direction when changing from A to B) correspond exactly to non-crossing nets of a common convex polyhedron. By envelope/parcel magic, it becomes easy to design many hinged dissections.

1 Introduction

Given two polygons A and B of equal area, a *dissection* is a decomposition of A into pieces that can be re-assembled (by translation and rotation) to form B. In a (chain) *hinged* dissection, the pieces are hinged together at their corners to form a chain, which can fold into both A and B, while maintaining connectivity between pieces at the hinge points. Many known hinged dissections are *reversible* (originally called *Dudeney dissection* [3]), meaning that the outside boundary of A goes inside of B after the reconfiguration, while the portion of the boundaries of the dissection inside of A become the exterior boundary of B. In particular, the hinges must all be on the boundary of both A and B. Other papers [4, 2] call the pair A, B of polygons *reversible.*

Without the reversibility restriction, Abbott et al. [1] showed that any two polygons of same area have a hinged dissection. Properties of reversible pairs of polygons were studied by Akiyama et al. [3, 4]. In a recent paper [2], it was shown that reversible pairs of polygons can be generated by unfolding a polyhedron using two non-crossing nets. The purpose of this paper is to show that this characterization is in some sense complete.

An *unfolding* of a polyhedron P cuts the surface of P using a *cut tree* T,[1] spanning all vertices of P, such that the cut surface $P \setminus T$ can be unfolded into the plane without overlap by opening all dihedral angles between the (possibly cut) faces. The planar polygon that results from this unfolding is called a *net* of P. Two trees T_1 and T_2 drawn on a surface are *non-crossing* if pairs of edges of T_1 and T_2 intersect only at common endpoints and, for any vertex v of both T_1 and T_2, the edges of T_1 (respectively, T_2) incident to v are contiguous in clockwise order around v. Two nets are non-crossing if their cut trees are non-crossing.

Lemma 1. *Let T_1, T_2 be non-crossing trees drawn on a polyhedron P, each of which spans all vertices of P. Then there is a cycle C passing through all vertices of P such that C separates the edges of T_1 from edges of T_2, i.e., the (closed) interior (yellow region) of C includes all edges of T_1 and the (closed) exterior of C includes all edges of T_2.*

We can now state our first characterization.

Theorem 2. *Two polygons A and B have a reversible hinged dissection if and only if A and B are two non-crossing nets of a common polyhedron.*

Proof sketch. To prove one direction, it suffices to glue both sides of the pieces of the dissection as they are glued in both A and B to obtain a polyhedral metric homeomorphic to a sphere, and note that this metric corresponds to the surface of some polyhedron [2]. In the other direction, we use Lemma 1 to define the sequence of hinges. Now the cut tree T_B of net B is completely contained in the net A and determines the dissection. □

Often times, reversible hinged dissections are also *monotone*, meaning that the turn angles at

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[1] For simplicity we assume that the edges of T are drawn using segments along the surface of P, and that vertices of degree 2 can be used in T to draw any polygonal path.

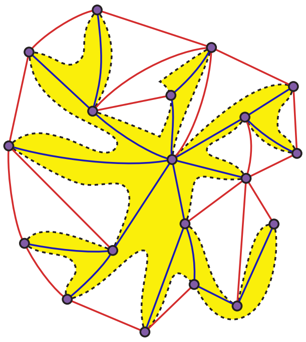

Figure 1: Example of Lemma 1. The edges of T_1, T_2 are colored blue, red, respectively.

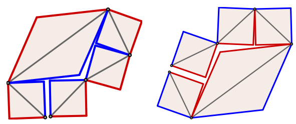

Figure 2: Reversible hinged dissection that is not monotone (or simple).

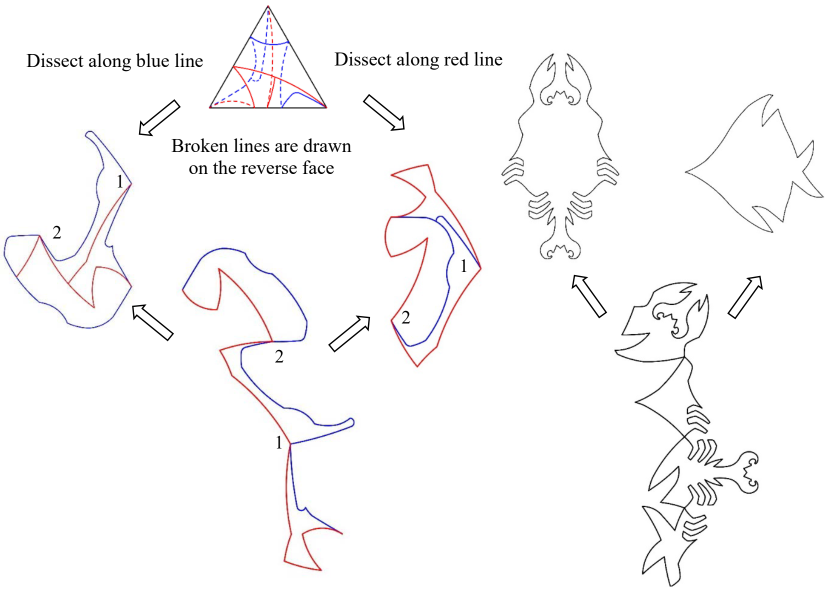

Figure 3: Two simple reversible hinged dissections found by our technique. Left: two non-crossing nets of a doubly covered triangle. Right: Lobster to fish.

all hinges in A increase to produce B. Figure 2 shows a hinged dissection that is reversible but not monotone. Monotone reversible hinged dissections also have a nice characterization:

Theorem 3. *Two polygons A and B have a monotone reversible hinged dissection if and only if A and B are two non-crossing nets of a common convex polyhedron.*

An interesting special case of a monotone reversible hinged dissection is when every hinge touches only its two adjacent pieces in both its A and B configurations, and thus A and B are only possible such configurations. We call these *simple* reversible hinged dissections. (For example, Figure 2 is not simple.)

Lemma 4. *Every simple reversible hinged dissection is monotone.*

Corollary 5. *If two polygons A and B have a simple reversible hinged dissection, then A and B are two non-crossing nets of a common convex polyhedron.*

Figure 3 shows two examples of hinged dissections resulting from these techniques. Historically, many hinged dissections (e.g., in [5]) have been designed by overlaying tessellations of the plane by shapes A and B. This connection to tiling is formalized by the results of this paper, combined with the characterization of shapes that tile the plane isohedrally as unfoldings of certain convex polyhedra [6].

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