

A Generalization of the Source Unfolding of Convex Polyhedra

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Dedicated to Ferran Hurtado on the occasion of his 60th birthday.

Abstract. We present a new method for unfolding a convex polyhedron into one piece without overlap, based on shortest paths to a convex curve on the polyhedron. Our “sun unfoldings” encompass source unfolding from a point, source unfolding from an open geodesic curve, and a variant of a recent method of Itoh, O’Rourke, and Vilcu.

1 Introduction

The easiest way to show that any convex polyhedron can be unfolded is via the *source unfolding* from a point s , where the polyhedron surface is cut at the *ridge tree* of points that have more than one shortest path to s , [10], or see [3]. The unfolding does not overlap because the shortest paths from s to every other point on the surface develop to straight lines radiating from s , forming a star-shaped unfolding. See Figure 1(b).

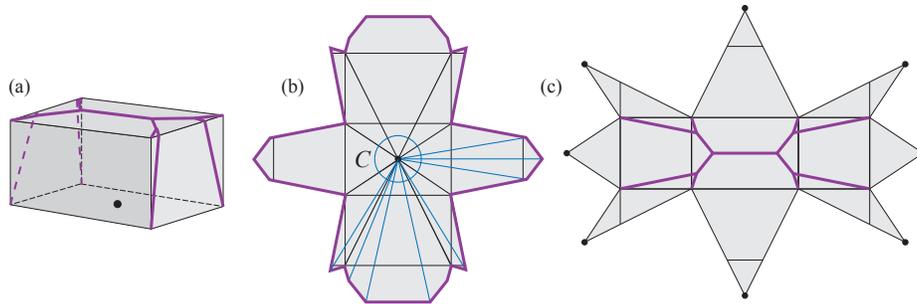


Fig. 1. [based on O’Rourke [9]] Unfolding a box from a point on the middle of the base: (b) source unfolding with some shortest paths shown. The source unfolding is the same as the sun unfolding relative to circle C . (c) star unfolding, with ridge tree shown.

Our main result is a generalized unfolding, called a *sun unfolding*, that preserves the property that shortest paths emanate in a radially monotone way,

although they no longer radiate from a point. We begin with an easy generalization where the point is replaced by a curve S that unfolds to a straight line segment (an *open geodesic curve*). Cutting at the ridge tree of points that have more than one shortest path to S produces an unfolding in which the shortest paths from S radiate from the unfolded S , so the unfolding does not overlap; see Figure 2.

For our general sun unfolding, the paths emanate radially, not from a point or a segment, but from a tree S , and the paths are not necessarily shortest paths from S . We define both S and the paths based on a convex curve C on the surface of the polyhedron. Let S be the ridge tree of C on the convex side and let R be the ridge tree of C on the other side. Let \mathcal{G} be the set of all shortest paths to C , where we glue together any paths that reach the same point of C from opposite sides. We prove that the paths emanate in a *radially monotone way* from the unfolded S , and hence that the polyhedron unfolds into a non-overlapping planar surface if we make the following cuts: cut R and, for every vertex v on the convex side of C , cut a shortest path from v to C and continue the cut across C , following a geodesic path, until reaching R . See Figure 3.

Our result generalizes source unfolding from a point or an open geodesic, by taking C to be the locus of points at distance ε from the source. See the curve C in Figure 1(b) and Figure 2(c). Our result is related to recent work of Itoh, O’Rourke, and Vîlcu on “star unfolding via a quasigeodesic loop” [6]. In the remainder of this section we discuss the relationship between these results.

Related Work

A *quasigeodesic loop* on the surface of a polyhedron is a closed polygonal curve such that every point on the curve except one has surface angle at most π on each side. Quasigeodesic loops are a special case of convex curves. Given a quasigeodesic loop Q , Itoh et al. [6] showed that cutting from every vertex to Q yields an unfolding of each half of the polyhedron, and furthermore that the two unfolded pieces can be joined together. See Figure 4. Our sun unfolding of this example is shown in Figure 4(c). The convex half unfolds the same way, but the non-convex half of the surface is cut into pieces and attached around the other half. Note that Itoh et al. call their unfolding a “star” unfolding from the quasigeodesic loop—they regard the quasigeodesic loop as the generalization of the point from which the star unfolding is defined. Our emphasis is different. We focus on the way that shortest paths emanate radially from the ridge tree S , so we regard the ridge tree as the generalization of the point from which the source unfolding is defined. Itoh et al. prove their unfolding result using Alexandrov’s work. We take a more self-contained approach of applying induction as the convex curve shrinks/expands on the surface of the polyhedron.

Itoh, O’Rourke, and Vîlcu also have an interesting alternative unfolding where the convex curve C remains connected (developing as a path) while S and R are cut [5, 7, 8]. This is possible only for special convex polygonal curves.

2 Source Unfolding From an Open Geodesic

In this section we generalize source unfolding from a point s to source unfolding from an *open geodesic curve*—a path on the surface of P that does not self-intersect, starts and ends at distinct points that are not vertices, and is a locally shortest path between those endpoints. See Figure 2. Although this generalization is quite easy to prove, it seems to be a new observation.

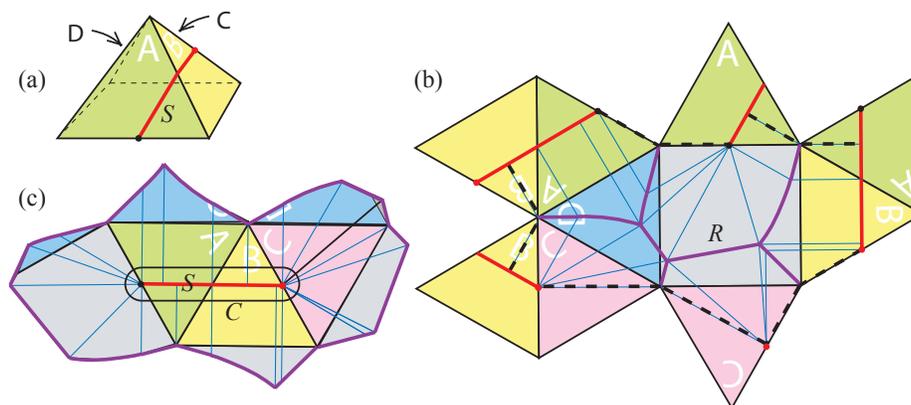


Fig. 2. Source unfolding from an open geodesic: (a) a pyramid with an open geodesic curve S crossing two faces; (b) the ridge tree R lies in two faces, the base and face D . (The dashed lines, together with segments of S delimit a “dual” unfolding where the paths are attached to R .); (c) the source unfolding showing paths emanating radially from the open geodesic, and showing the convex curve C relevant to sun unfolding.

Lemma 1. *Let S be an open geodesic curve on the surface of a convex polyhedron P , and let R be the ridge tree of points on P that have more than one shortest path to S . Then cutting along R produces an unfolding of P that does not overlap.*

Proof. We will assume that R is a tree that includes all vertices of P —this follows from our general result below, or can be proved as Sharir and Schorr [10] do for the case where S is a point. Therefore R unfolds P to a planar surface. Because S is a geodesic, it unfolds to a straight line segment. Consider any point p on P that is not in R . Then there is a unique shortest path from p to S . We claim that this path reaches S in one of two ways: (1) it reaches an interior point of S and makes a right angle with S ; or (2) it reaches an endpoint of S and makes an angle between $\pi/2$ and $3\pi/2$ with S . The reason is that a path that reaches S in any other way can be shortened.

Observe that these shortest paths spread out in a radially monotone way from the unfolded S . See Figure 2. Thus we have an unfolding of P that does not overlap. \square

3 Sun Unfolding

We define *sun unfolding* of a convex polyhedron P relative to a closed convex curve C on P . We will prove our unfolding result only for curves composed of a finite number of line segments and circular arcs, but will discuss more general convex curves. The curve C splits P into two “halves”, the convex or *interior* side C_I , and the *exterior* side C_E . For a point c on C (notated $c \in C$), let $\alpha_I(c)$ be the surface angle of C_I between the left and right tangents at c , and let $\alpha_E(c)$ be the surface angle of C_E between those tangents. Then $\alpha_I(c) + \alpha_E(c) \leq 2\pi$, with equality unless c is a vertex of P . Also, $\alpha_I(c) \leq \pi$. A point c with $\alpha_I(c) < \pi$ is called an *internal corner* of C . A point $c \in C$ with $\alpha_E(c) < \pi$ is called an *external convex corner* of C . If a point $c \in C$ is a vertex then it is an internal corner or an external convex corner (or both); and if c is not a vertex, then it is not an external convex corner. See Figure 5.

The *ridge tree* (a.k.a. “cut locus”) in C_I [or C_E] is the closure of the set of points that have more than one shortest path to C . Let S be the ridge tree of C in C_I , and let R be the ridge tree in C_E . Among all the shortest paths from points of C_I to C , let \mathcal{G}_I be the maximal ones. Among all the shortest paths from points of C_E to C , let \mathcal{G}_E be the maximal ones. If $c \in C$ has $\alpha_I(c) = \alpha_E(c) = \pi$, then we concatenate together the unique paths of \mathcal{G}_I and \mathcal{G}_E that are incident to c . Let \mathcal{G} be the resulting set of paths, together with any leftover paths of \mathcal{G}_I and any leftover paths of \mathcal{G}_E . For example, a leftover path of \mathcal{G}_I reaches c_3 in Figure 5; observe that c_3 is in R . The figure also shows examples of leftover paths of \mathcal{G}_E reaching points c_2 and c_4 ; observe that c_2 and c_4 are in S .

Lemma 2. *Both R and S are trees. Every vertex of P lies in R or S (or both). Every internal corner of C is a leaf of S . Every external convex corner of C is a leaf of R . Every path of \mathcal{G} goes from S to R and includes a point of C . The surface of P is covered by S , R , and \mathcal{G} . Furthermore, any point not on S or R is in a unique path of \mathcal{G} .*

The proof of the lemma is in Section 4.

Let v be a vertex of P . If v is not in R , then it is in S , and we let $\gamma(v)$ be a path of \mathcal{G} incident to v . The choice of $\gamma(v)$ is not unique in general, but we fix one $\gamma(v)$. Observe that each $\gamma(v)$ is a path from v to R , consisting of a shortest path from v to C possibly continued geodesically to R . We define *sun cuts* with respect to C to consist of R and the paths $\gamma(v)$, for v a vertex of P in $C_I \cup C$. Note that a vertex on C may be a leaf of R , in which case $\gamma(v)$ has length 0.

Theorem 1. *Let C be a closed convex curve on the surface of a convex polyhedron P , such that C is composed of a finite number of line segments and circular arcs. Then sun cuts with respect to C unfold the surface of P into the plane without overlap.*

To prove the theorem, we first show that the sun cuts form a tree that reaches all vertices of P —hence the surface unfolds to the plane. To show that the unfolded surface does not overlap, we prove, by shrinking C and applying

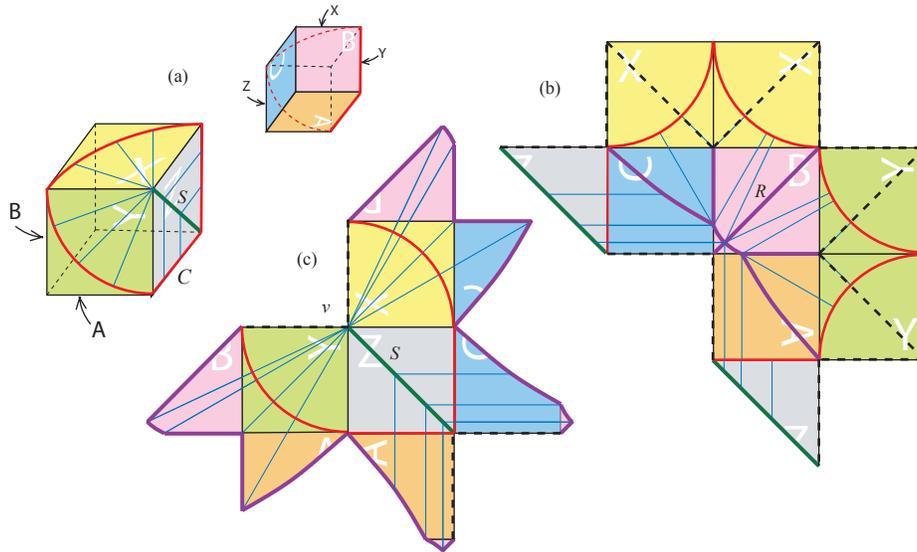


Fig. 3. Sun unfolding with respect to a convex curve: (a) a cube with a convex curve C and the ridge tree S on the convex side; (b) the ridge tree R on the non-convex side of C . (The dashed lines delimit a “dual” unfolding where the paths are attached to R .); (c) the sun unfolding showing paths emanating radially from S . Note that the two vertices on S require cuts $\gamma(\cdot)$ to R , shown with dashed lines.

induction, that S unfolds without overlap and that the paths of \mathcal{G} emanate from the unfolded S in a *radially monotone way*, defined as follows. Make a tour clockwise around the unfolded S , travelling in the plane an infinitesimal distance away from the unfolded S . See Figure 4(c). Parts of the tour are off the unfolded surface P . In particular, whenever a cut reaches S , there will be a gap in the unfolding. Apart from the gaps, at any point of the tour we are at a point p of $P - (S \cup R)$, so by Lemma 2 there is a unique path $\gamma(p) \in \mathcal{G}$ containing p and extending to R . Extend this path to a ray, and let $f(p)$ be the corresponding point on the circle at infinity. We say that \mathcal{G} *emanates from the unfolded S in a radially monotone way* if, as p tours clockwise around S , $f(p)$ progresses clockwise around the circle at infinity, i.e., if $p'' > p' > p$ along the tour then $f(p'') \geq f(p') \geq f(p)$ clockwise around the circle at infinity.

4 Sun Unfolding: Proofs

We begin with a section about the local properties of shortest paths to C . Following that is a section on properties of the ridge trees, and in the final section we prove that the sun unfolding does not overlap.

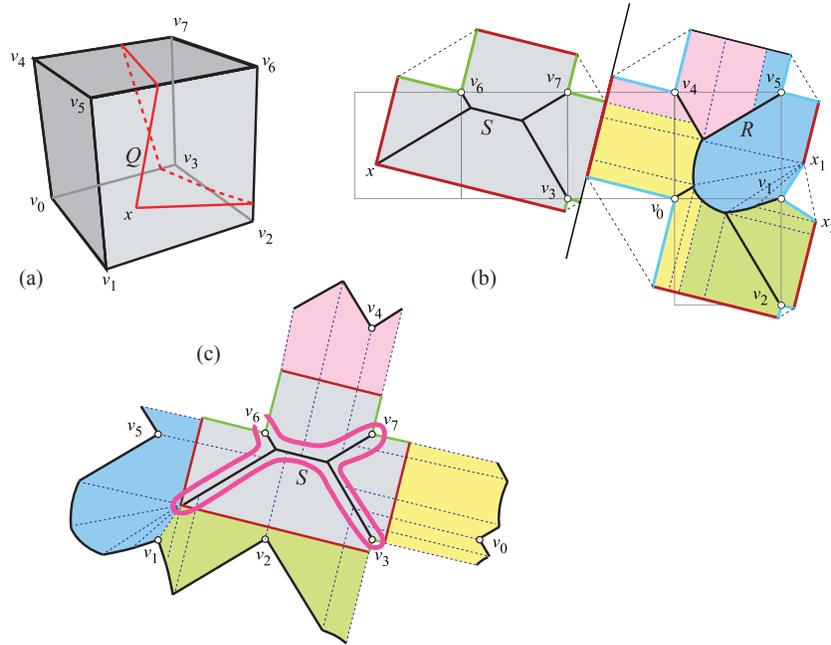


Fig. 4. The sun unfolding with respect to a geodesic loop Q on a cube, based on an example from Fig. 1 of Itoh et al. [6]: (a) the cube and the quasigeodesic loop Q ; (b) the ridge trees S and R on the two sides of the curve Q (superimposed on the unfolding from [6]); (c) the sun unfolding with respect to Q , showing a tour around S and the paths emanating in a radially monotone way from S .

4.1 Structure of Shortest Paths

Claim. Two shortest paths do not cross. [6, Lemma 2]. A shortest path reaches C at an angle greater than or equal to $\pi/2$.

We need the second statement only for the case where C is composed of a finite set of straight line segments and circular arcs, where it seems obvious (a path reaching C at an angle less than $\pi/2$ can be locally shortened). A more general case is proved in [4].

As a consequence, we make the following observations about how shortest paths behave with respect to corners and reflex points of a convex curve C . See Figure 5. Let c be a point of C , and let s be a side, i.e., $s = I$ or E .

- If $\alpha_s(c) = \pi$ then there is exactly one path in \mathcal{G}_s to point c .
- If $\alpha_s(c) < \pi$ then there is no path of \mathcal{G}_s to point c .
- If $\alpha_E(c) > \pi$ then there is a wedge of paths of \mathcal{G}_E to point c . In this case, $\alpha_I(c)$ must be less than π .

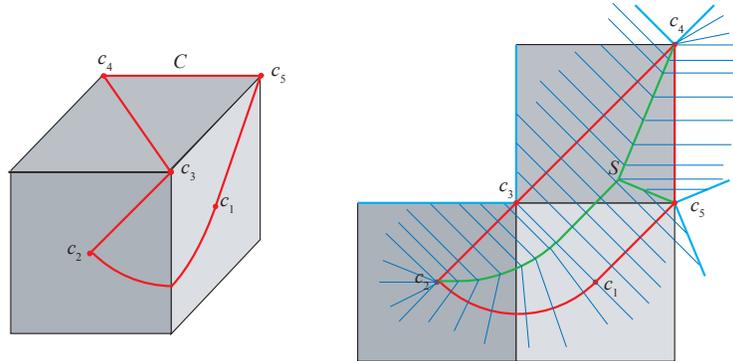


Fig. 5. A convex curve C on the surface of a cube (*left*), and part of the sun unfolding with respect to C (*right*) showing the ridge tree S and some of the paths of \mathcal{G} . The points c_i show some possible vertex/corner/reflex point combinations.

4.2 Structure of the Ridge Trees

We believe that the sun unfolding works for any convex curve C , but our proof technique requires that the ridge trees of the curve C are trees, not only in the sense of being connected and acyclic, but also in the sense of having a finite number of nodes. Even for a closed curve in the plane, the ridge tree (which is then the *medial axis*) need not be finite: Choi, Choi and Moon [2] give an example of a smooth curve (in fact C^∞) whose medial axis has an infinite number of leaves. Their curve alternates infinitely often between convex and concave parts, but it is also possible to give an example of a convex (non-smooth) planar curve whose ridge tree has infinitely many leaves.

We will prove that the ridge trees are finite, and that the sun unfolding works, for any convex curve C composed of a finite number of line segments and circular arcs. We will discuss weaker conditions that probably suffice.

Our proofs use a “wavefront expansion” where we offset the curve C by distance $d \geq 0$, forming curve C_d . As d increases (and C “shrinks”) we stop at “events” where the curve changes in an essential way. The ridge tree of C consists of the ridge tree of C_d together with the portion of the ridge tree in the “band” between C and C_d , so we obtain our results by induction as long as the number of events is finite. This is why we need a finite ridge tree.

Shortest paths reach C along normals to the curve, which are maintained as C shrinks. At a point of C with surface angle less than π (a *corner*), the normals to the curve at either side of the point meet and form an edge of the ridge tree. As C shrinks, the corner traces out the edge of the ridge tree. At a point of C with surface angle greater than π (a “reflex” point), there is a wedge of normals reaching the point. For any $d > 0$ the shrunken curve C_d has a circular arc in place of the point. Thus reflex points vanish immediately. We prove below that no reflex points arise during the shrinking process.

Therefore the general structure is that C is a curve with surface angle less than or equal to π at every point. The corners (where the surface angle is less than π) trace out the edges of the ridge tree as C shrinks. For our application, we have a convex curve that we shrink both inside and outside. With respect to the inside, all points other than corners have positive curvature; with respect to the other side, for any $d > 0$, all points of C_d other than corners have negative curvature. Furthermore, for the special case of curves composed of a finite number of line segments and circular arcs, there are finitely many corners.

Events. As C shrinks, the possible events are as follows. Note that more than one event may happen simultaneously, but we claim that we can handle them one at a time.

Closing Event. The curve closes up and vanishes. It may vanish at a point, or more generally, at a tree, which forms part of the ridge tree. In case C is the inside or outside of a convex curve, C can only collapse to a point or a straight line segment.

Vertex Event. The curve encounters a vertex v of P . See Figure 6. As the curve shrinks past v the surface angle of the curve at v decreases by the angular defect of v . The surface angle was at most π before hitting v , and is strictly less than π afterwards. Thus there is a corner at v after the event. It is a new corner except in the degenerate situation when an existing corner hits v . An edge of the ridge tree grows at the corner.

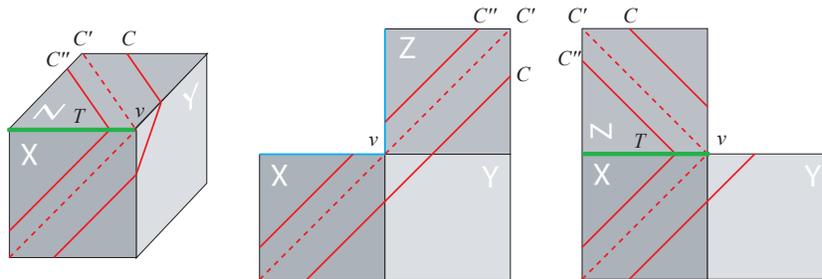


Fig. 6. A vertex event on a cube. The curve C shrinks to C' , forming a corner at vertex v , and then shrinks to C'' , tracing out an edge of the ridge tree T .

New Corner Event. See Figure 7(a). In the general case a new corner appears at a convex point of the curve where the the radius of the *osculating circle* becomes 0 (i.e. the curvature approaches positive infinity). In the special case of curves composed of line segments and circular arcs, a new corner appears when a circular arc shrinks to a point p . No new corners appear if C is the outside of a convex curve.

Corner Merge Event. See Figure 7(b). Two edges of the ridge tree meet and a new edge forms. In degenerate cases, it is possible that more than two edges of the ridge tree meet.

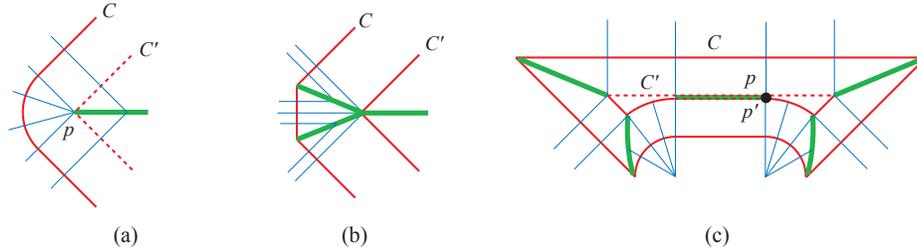


Fig. 7. Events: (a) a new corner event; (b) a corner merge event; (c) a pinch event. Thick lines indicate the ridge tree, and thin lines indicate paths of \mathcal{G} .

Pinch Event. The curve meets itself, but does not vanish. Figure 7(c) shows the curve meeting itself along a segment. Suppose point p becomes coincident with point p' , but no part of the curve from p to p' (traversed clockwise, say) becomes coincident.

Lemma 3. *In the closed curve from p to p' , point $p = p'$ is a corner with surface angle 0 and turn π . Furthermore, there must be negative curvature arbitrarily close to p or p' (or both).*

Proof. Every point on the curve has surface angle at most π . Thus the surface angle of the original curve must be exactly π at p and at p' (because if one were less than π the other would have to be greater than π). This implies that after the collapse, point $p = p'$ has surface angle 0 and therefore turn π .

Now consider the curvature. If p , say, has positive curvature, then p' must have negative curvature, which completes the proof. So suppose some ε -interval of the curve from p has 0 curvature, and some ε' -interval of the curve from p' has 0 curvature. Then the curves are line segments in those intervals, and must be coincident within the smaller interval. Note that for the case of curves composed of a finite number of line segments and circular arcs, the neighbourhood argument is unnecessary— p or p' must have negative curvature. \square

Therefore, for a convex curve, a pinch event can only happen on the outside, and cannot involve three or more points of C meeting at the same point. The curve may become coincident along a segment, in which case the claim applies to both endpoints of the common segment in the two resulting closed curves.

Ridge trees. We have a convex curve C that shrinks to the inside, forming ridge tree S in C_I , and to the outside, forming ridge tree R in C_E . On the outside, new corner events do not occur because there are no convex points apart from the

corners. We claim that the ridge tree is therefore a [finite] tree—see Lemma 4 below. On the inside, Lemma 3 implies that pinch events do not occur, and, so long as new corner events happen only a finite number of times, we claim that the ridge tree is a [finite] tree—see Lemma 5 below. We will deal explicitly only with the case where C is composed of a finite number of line segments and circular arcs, which guarantees a finite number of new corner events.

Lemma 4. *R is a [finite] tree. R contains all the vertices of P that lie in C_E , and R has a leaf at each external convex corner of C .*

We prove the lemma only for the case where C is a convex curve composed of a finite number of line segments and circular arcs, but we believe it holds for convex curves generally.

Proof. For the proof we will change our frame of reference with respect to inside/outside. We shrink C into C_E , and will now refer to that as “inside”. We will also say “corner” rather than “external convex corner.”

We shrink the curve and argue by induction on $2n + c$ where n is the number of vertices lying inside C , and c is the number of corners of C . We use the property that C is composed of a finite number of line segments and circular arcs of negative curvature joined at points with surface angle at most π , and note that this holds after each event.

Shrink C to C' at the next event. As noted above in the general case, new corner events only occur when the curvature at some point approaches positive infinity. Because our curve has negative curvature except at the corners, there are no new corner events. We are left with four possible events:

Closing event. C' is either a point or a line segment, since two circular arcs of negative curvature cannot collapse together. C' forms its own ridge tree, and the portions of the ridge tree edges between C and C' attach to it, which gives the desired result for C .

Vertex event. As noted above, there is a corner at vertex v after the event, and it is a new corner except in the degenerate situation when an existing corner hits v . If the point of the curve that hits v is an internal point of a line segment or circular arc, then the segment or arc splits in two. The properties of the curve are maintained. Observe that n goes down by 1 and c goes up by at most 1. Therefore the quantity $2n + c$ goes down, and by induction the ridge tree of C' is a tree that contains each vertex interior to C' and has a leaf at each corner of C' . The portions of the ridge tree edges between C and C' attach to the leaves of the ridge tree at the corners of C' , which gives the desired result for C .

Corner merge event. In this case c goes down by at least 1 and n stays the same. The quantity $2n + c$ goes down and the result follows by induction.

Pinch event. By Lemma 3, C' consists of two closed curves C'_1 and C'_2 . The vertices interior to C are partitioned between C'_1 and C'_2 , as are the corners of C . Also, by Lemma 3, each of C'_1 and C'_2 has one new corner, say at points p_1 and p_2 , respectively.

We first argue that we can apply induction to C'_1 and C'_2 . If C'_i , $i = 1$ or $i = 2$ has fewer than n vertices in its interior, then the quantity $2n + c$ goes

down and we can apply induction. Suppose C'_1 has n vertices in its interior. Then C'_2 has no vertices in its interior, which implies that it lives in the plane and therefore has total turn 2π . Furthermore, by Lemma 3, the new corner p_2 of C'_2 has turn π and at least one part of the curve incident to p_2 has negative curvature. Since all points except corners have non-positive curvature, the other corners of C'_2 must have turn sum greater than π , and therefore there must be at least two other corners. This implies that C'_1 has strictly fewer corners than C . Therefore we can apply induction to C'_1 .

By induction, the ridge trees of C'_1 and C'_2 are trees and contain all the vertices of P that lie inside C . Furthermore, the ridge trees have leaves at p_1 and p_2 , respectively. If $p_1 = p_2$ (i.e., the pinch event occurred at a single point) then the two ridge trees join to form the ridge tree of C . Otherwise, the pinch event occurred along a segment, and the segment plus the two ridge trees form the ridge tree of C . \square

Lemma 5. *S is a [finite] tree. S contains all the vertices of P that lie in C_I , and S has a leaf at each internal corner of C .*

We prove the lemma only for the case where C is a convex curve composed of a finite number of line segments and circular arcs, but we believe that the lemma holds more generally for a convex curve so long as the number of leaves of S is finite, and that a natural sufficient condition for this is that there are a finite number of points of maximal curvature along C .

Proof. We are focused on the inside of C , so we will just say “corner” rather than “internal corner”. The proof is by induction on $4n + 2a + c$ where n is the number of vertices lying in the interior of C , c is the number of corners of C , and a is the number of circular arcs of C . We prove in addition that the curve is always convex and composed of line segments and circular arcs.

Shrink C to C' at the next event. By Lemma 3, a pinch event can only occur when the curve has a point of negative curvature, and therefore pinch events do not occur when C is convex. We consider each of the possible events:

Closing event. The argument is the same as in Lemma 4.

Vertex event. The argument about the ridge tree is the same as in Lemma 4. Convexity is maintained. Observe that n goes down by 1, c goes up by at most 1, and a goes up by at most 1. Therefore the quantity $4n + 2a + c$ goes down, and by induction the ridge tree of C' is a tree that contains each vertex interior to C' and has a leaf at each corner of C' . The portions of the ridge tree edges between C and C' attach to the leaves of the ridge tree at the corners of C' , which gives the desired result for C .

New corner event. This occurs only when one of the circular arcs of C shrinks to a point. The point becomes a corner of C' . Convexity is maintained. Observe that c goes up by 1, a goes down by 1, and n is unchanged. Therefore the quantity $4n + 2a + c$ goes down, and by induction the ridge tree of C' is a tree that contains each vertex interior to C' and has a leaf at each corner of C' . Adding back the portions of ridge tree edges between C and C' gives the result for C .

Corner Merge Event. In this case c goes down by at least 1, a does not go up, and n stays the same. The quantity $4n + 2a + c$ goes down and the result follows by induction. \square

Having established the structure of R and S , we now wrap up the proof of Lemma 2.

Proof (of Lemma 2). By Lemma 5, S is a tree that includes every vertex of P that lies in C_I and every internal corner of C . By Lemma 4, R is a tree that includes every vertex of P that lies in C_E and every external convex corner of C . It remains to show that every vertex of P lies in R or S , or both. Vertices in C_I and C_E are in the respective trees. Consider a vertex v of P that lies on C . Since the surface angle at v is less than 2π , v must be an internal corner or an external convex corner. Therefore C is in S or R , or both.

To complete the proof, we must establish the properties of \mathcal{G} . By definition, every path γ of \mathcal{G} includes a point of C . If γ was formed by concatenating together a path of \mathcal{G}_I and \mathcal{G}_E then γ goes from S to R . If γ is a leftover path of \mathcal{G}_I then it reaches a point $c \in C$ with $\alpha_E(c) < \pi$, so c is a convex external corner of C , and c is in R . If γ is a leftover path of \mathcal{G}_E then it reaches a point $c \in C$ with $\alpha_I(c) < \pi$, so c is an internal corner of C and c is in S . Finally, it is clear that the surface of P is covered by S , R , and \mathcal{G} , and that any point not on S or R is in a unique path of \mathcal{G} . \square

4.3 Non-overlap of the Sun Unfolding

Lemma 6. *The sun cuts with respect to a closed convex curve C form a tree incident to every vertex of P .*

Proof. By definition, the sun cuts consist of R together with paths $\gamma(v)$ for v a vertex of P in $C_I \cup C$. By Lemma 4, R is a tree. Adding the paths $\gamma(v)$ still gives a tree, because every such path contains exactly one point of R and no two such paths intersect—this is because the portions of the paths in C_I do not intersect, the points where the paths reach C are distinct, and thus the portions of the paths in C_E do not intersect (except possibly at R). Finally, by Lemma 4, R includes any vertex interior to C_E and, by construction, we add cuts reaching every vertex in $C_I \cup C$. \square

As discussed in Section 3, we will prove that the sun cuts unfold P without overlap by proving that the paths \mathcal{G} emanate from the unfolded S in a radially monotone way. We will use an equivalent formulation of radial monotonicity. Let $\bar{\mathcal{G}}$ be the infinite rays in the plane that extend paths of \mathcal{G} from their point of origin on the unfolded S . Radial monotonicity is equivalent to the property that two rays of $\bar{\mathcal{G}}$ do not intersect except at a common point of origin. The main work, in Lemma 7 below, is to prove this for the interior of C_I and for the rays $\bar{\mathcal{G}}_I$ that originate from points of S interior to C_I . Note that the sun cuts, restricted to C_I , consist of cuts from every vertex inside C_I along a shortest path to C . Itoh et al. [6] call this the “star unfolding” and prove that it develops

without overlap for quasigeodesic loops. We need more general convex curves and we need the stronger result about radial monotonicity.

Lemma 7. *Two rays of $\bar{\mathcal{G}}_I$ do not intersect except at a common point of origin. Consequently, the sun cuts unfold the interior of C_I without overlap.*

Proof. We prove the result by induction as C shrinks. We only treat the case where C consists of a finite number of line segments and circular arcs. The proof follows the same structure as the proof of Lemma 5. In particular, the induction is on the quantity $4n + 2a + c$, and we will not repeat the arguments about why we can apply induction at an event.

Recall that curve C shrinks to curve C' at the next event, and that there are no pinch events because C is convex. The ridge tree S consists of the ridge tree S' of C' together with the part of S that lies in the “band” between C and C' . Let $\bar{\mathcal{G}}'$ denote the rays of $\bar{\mathcal{G}}$ that are incident to S' . Let U' denote the unfolding of the interior of C' with the extended rays $\bar{\mathcal{G}}'$. Let U denote the analogous structure for C . By induction, we know that U' does not overlap itself. We want to prove the same for U .

To get U from U' we need to add the pieces of S that lie in the band between C and C' and the rays of $\bar{\mathcal{G}}$ that originate from these new pieces of S . Let s_c be the portion of a ridge tree edge that is traced out by a corner c of C as it shrinks to corner c' of C' . See Figure 8. Let $\bar{\mathcal{G}}_c$ be the rays of $\bar{\mathcal{G}}$ that originate at points of s_c . The rays of $\bar{\mathcal{G}}_c$ that originate from the point c' are the two normals to C' at c' . Because C is convex, the other rays of $\bar{\mathcal{G}}_c$ unfold to lie in a wedge bounded by these normals. We need to show that this wedge fits into the unfolding U' . Note that c' is locally flat (we consider vertex events below). Also note that, because c' is a corner of C' , no shortest path from the interior of C' reaches C' at c' . This means that no sun cut reaches c' , and hence a neighbourhood of c' is intact in the unfolding U' . Therefore U' has a wedge-shaped gap at c' that exactly accommodates $\bar{\mathcal{G}}_c$ (including the segment s_c).

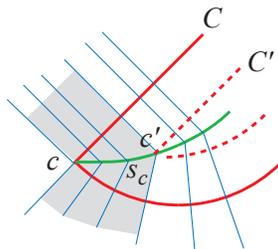


Fig. 8. A piece, s_c , of the ridge tree traced out by corner c of C as it shrinks to corner c' of C' . The shaded area indicates the rays of $\bar{\mathcal{G}}_c$.

The situations at a new corner event and a corner merge event are similar. Refer back to Figure 7(a) and (b). In both cases the event results in a corner c'

of C' . There is a piece s_c of $S - S'$ attached to c' (possibly just the point c'), and some rays of $\bar{\mathcal{G}}$ originate from points of s_c . By local planarity, these fit into the gap at c' in the unfolding U' .

At the closing event, the convex curve C collapses to a point or line segment, which becomes the ridge tree. The paths of $\bar{\mathcal{G}}$ radiate outward from the ridge tree without intersections.

Finally, we consider a vertex event. Suppose C' goes through vertex v of P . Then the sun cuts include a shortest path from v to C . The unfolding U' has a gap at v with angle equal to the angular deficit at v . The unfolding U has no extra surface in the neighbourhood of v , and is therefore the same as U' in that neighbourhood. \square

Proof (of Theorem 1). We prove that sun cuts unfold P without overlap by proving that no two rays of $\bar{\mathcal{G}}$ intersect except at a common point of origin. Lemma 7 proves this for $\bar{\mathcal{G}}_I$. It remains to add the rays of $\bar{\mathcal{G}}$ originating from points of S that lie on C .

Let c be a leaf of S lying on C that has paths of \mathcal{G} incident to it. Then c is a point with $\alpha_I(c) < \pi$ and $\alpha_E(c) > \pi$. As in the proof of Lemma 7, a neighbourhood of c in C_I is intact in the unfolding, and has a wedge-shaped gap large enough for the wedge of rays of $\bar{\mathcal{G}}$ originating at c . In case c is a vertex, one of the rays of $\bar{\mathcal{G}}$ incident to c is in the set of sun cuts, so the rays of $\bar{\mathcal{G}}$ originating at c are split into two wedges with a gap between them equal to the angular defect of the vertex. \square

5 Unified Unfolding

The sun unfolding generalizes one of the basic unfolding methods for convex polyhedra, namely the source unfolding from a point. The other basic unfolding method is the *star unfolding* from a point s , where the polyhedron surface is cut along a shortest path from every vertex to s [1]. See Figure 1(c). This is dual to the source unfolding in that the shortest paths are attached in one case to s (for source unfolding) and in the other case to the ridge tree (for star unfolding). The dual of our sun unfolding would be to attach the paths of \mathcal{G} to the ridge tree R and cut the ridge tree S and paths of \mathcal{G} from vertices to S . See for example Figure 2(b) and Figure 3(b). We conjecture that this unfolds without overlap. In the general dual case the paths \mathcal{G} do not emanate in a radially monotone way from R , so a new proof technique will be needed. A first step would be to prove that the star unfolding from an open geodesic unfolds without overlap, i.e. to prove the dual version of Lemma 1.

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