Tangled Tangles

Erik D. Demaine[1], Martin L. Demaine[1], Adam Hesterberg[2], Quanquan Liu[1], Ron Taylor[3], and Ryuhei Uehara[4]

[1]Computer Science and Artificial Intelligence Lab, Massachusetts Institute of Technology, Cambridge, MA, USA
{edemaine,mdemaine,quanquan}@mit.edu
[2]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA
achester@mit.edu
[3]Berry College, Mount Berry, GA, USA
rtaylor@berry.edu
[4]Japan Advanced Institute of Science and Technology, Ishikawa, Japan
uehara@jaist.ac.jp

1 Introduction

The *Tangle* toy [Zaw15, Zaw85] is a topological manipulation toy that can be twisted and turned in a variety of different ways, producing different geometric configurations. Some of these configurations lie in 3D space while others may be flattened into planar shapes. The toy consists of several curved, quarter-circle pieces fit together at rotational/twist *joints*. Each quarter-circle piece can be rotated about either of the two joints that connect it to its two neighboring pieces. Fig. 1 shows a couple of Tangle toys that can be physically twisted into many 3D configurations. See [Zaw15] for more information and demonstrations of the toy.

Figure 1: Two Tangle toys. Photo by Quanquan Liu, 2015.

More precisely, an n*-Tangle* consists of n quarter-circle *links* connected at n *joints* in a closed loop.[1] Tangles can move into many configurations by rotating/twisting the joints along the axis

[1]Previous literature have also called the quarter-circles "pieces" or "segments." Here, we choose to use "links" for greater specificity (when characterizing Tangle structures as fixed-angle linkages) and clarity.

of the two incident links (which must meet at a 180° angle). Fig. 2 shows an example of such an axis of rotation that joints may be rotated along. The links connected to the joint in Fig. 2 can be twisted clockwise or counterclockwise about the axis as shown by the two arrows. While Tangle configurations usually lie in 3D space, we focus in this paper on *planar* Tangle configurations, or Tangle configurations that can be flattened on a flat surface.

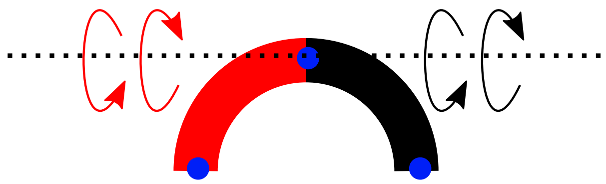

Figure 2: Blue dots represent joints. The axis is represented by the dotted line. Arrows show that both the red and black link can be rotated clockwise and counterclockwise about the axis.

Previous research into planar Tangle configurations [Cha03, Fle00] makes an analogy between Tangle and cell-growth problems involving polyominoes. An n-omino is composed of n squares of equal size such that every square is connected to the structure via incident edges. A well-known problem involving polyominoes is how many distinct *free* (cannot be transformed into each other via translations, rotations, or reflections) n-ominoes are there for $n = 1, 2, 3 \ldots$. Various previous research have succeeded in enumerating the number of free n-polyominoes up to $n = 28$ [Red81, Mer90, eS15]. Fig. 3 shows the 2 possible free configurations for the tromino.

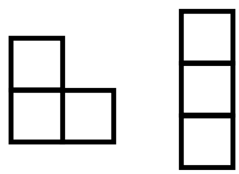

Figure 3: There are only two possible distinct free trominoes.

Using the analogy to polyominoes where a Tangle link represents a polyomino cell, [Cha03, Fle00] pose two questions for planar Tangle configurations. First, what is the number of distinct planar n-Tangles for $n = 4i$ where $i = 1, 2, 3, \ldots$ (called the "Tanglegram sequence")? In other words, given a Tangle toy with n links, what is the number of distinct planar Tangles that can be formed? Second, can any planar n-Tangle be transformed into any other planar n-Tangle according to certain moves described in Section 3 below? It was conjectured but not proven that this is possible.

The problem of determining whether all planar configurations can be reconfigured into each other using allowable moves is known as *flat-state connectivity* [ADD+02] of *linkages*. Recall that a Tangle toy is an example of a linkage that is composed of links and joints. One can think about the links in the linkage as "edges" and the joints as "vertices"; just as how vertices connect edges, joints connect links. Thus far, the study of flat-state connectivity has focused on *fixed-angle linkages*, where each link has an assigned fixed length and each vertex has an assigned fixed angle (i.e. the angle of incidence between two incident links is fixed). A *flat state* of such a linkage is an embedding of the linkage into $\mathbb{R}^2$. A linkage is *flat-state connected* if any two flat states of the linkage can be reconfigured into each other using a sequence of dihedral motions without self-intersections. Otherwise, the linkage is *flat-state disconnected*. All open chains with no acute angles, and all closed orthogonal unit chains, are flat-state connected, while open chains with 180° edge spins and graphs (as well as partially rigid trees) are flat-state disconnected [ADD+02]. For more details regarding these linkages, please refer to [ADD+02]. Closed orthogonal unit fixed-angle chains (chains that have unit length edges and 90° angles of incidence between edges) move essentially like Tangles (viewing each quarter-circle link as a 90° corner between two half edges), so their flat-state connectivity means that there are (complex, three-dimensional) moves between any two planar Tangle configurations of the same length.

The previous study of reachable configurations of Tangles consider a set of moves called x- and Ω-rotations [Cha03, Fle00]. In this paper, we generalize these moves into two broad categories, reflections and translations, that allow for a larger set of possible moves. In particular, our reflection moves involve rotating one chain of the Tangle by 180° around the rest, effectively reflecting the former. Such reflections over an axis (such as "flipturns", "Erdős pocket flips", and "pivots") has been studied by previous work in transforming planar polygons [ABB+07, ACD+02, DGOT08, ADE+01]. The purpose of such moves is to simplify complex moves involving many edge flips and rotations into simpler, more "local" moves. The reflection and translation moves used in our paper together encompass all possible edge flips around any two joints in a Tangle; thus, they seem natural to use as simplifications of more complex Tangle moves. More details of these moves as well as their relation to the previous x- and Ω-rotations can be found in Section 3.

Our results show that Tangle configurations are flat-state disconnected under even our general reflection and translation moves, in particular, disproving a conjecture of [Cha03]. This result provides an example of nontrivial flat-state disconnectedness. Planar Tangle configurations are natural examples of flat-state configurations obtained using a set of "local" moves around two joints. We show examples of planar Tangle configurations that have no moves whatsoever, as well as examples that have a few moves but cannot escape a small neighborhood of configurations.

In addition to our results on Tangle flat-state connectivity, we present two different Tangle fonts. This is a continuation of a study on mathematical typefaces based on computational geometry, as surveyed in [DD15a]. The two Tangle fonts were created from 52- and 56-Tangles.

We define some notations and conventions we follow in this paper in Section 2. Then, we describe the two classes of moves we considered in evaluating planar Tangles and their reachable configurations in Section 3. In Section 4, we present some examples of planar Tangles that are locked or rigid under our specified set of moves. In Section 5, we present the two Tangle fonts. Finally, in Section 6, we conclude with some open questions.

2 Definitions

A Tangle link can have two possible orientations with respect to the body of the structure, *convex* or *reflex*; see Fig. 4.

A *face* in a planar Tangle configuration consists of a set of convex links. Two faces are *tangential* if they are connected by reflex links. Fig. 5 shows some examples of faces. It was previously shown that an n-Tangle can form planar Tangle configurations if and only if n is a multiple of 4 [Fle00].

Using this definition of faces, we can further define the dual graph representation of a planar Tangle configuration to be a graph consisting of a vertex for each face of the configuration with an edge connecting each pair of tangential faces (see Fig. 5). This definition is analogous to the graph representation definition given by [Tay15].

This *dual graph* representation of planar Tangle configurations is useful in some proofs in the

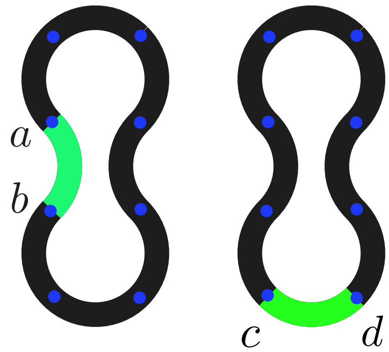

Figure 4: Reflex (a, b) and convex (c, d) links.

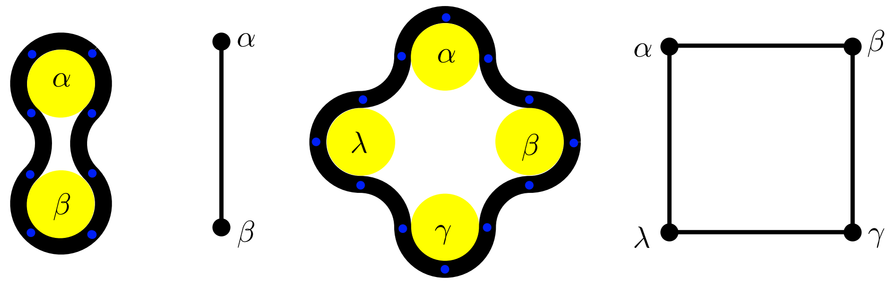

Figure 5: Dual graphs of planar Tangle configurations. α, β, γ, and λ label faces connected to other faces by reflex links.

later sections. Furthermore, this dual graph representation is used by our planar Tangle moves enumerator to enable users to easily create arbitrarily shaped planar configurations [DD15b].

3 Tangle Moves

In this section, we describe the set of legal moves that can be performed on any planar Tangle configuration. We categorize these moves broadly as *translation* and *reflection* moves, with the distinguishing factor being that translation moves are asymmetric while reflection moves can be performed along a reflection axis.

3.1 Reflections

Reflection moves are performed over a linear *reflection axis*, which consists of a line through two joints of the Tangle. We call these two joints the *reflection joints*. To perform a reflection, one of the two parts of the Tangle separated by the reflection joints is rotated 180° clockwise or counterclockwise around the reflection axis. In fact, the previously mentioned x- and Ω-rotations [Cha03] are reflection moves[2].

The reflection move may only be made if

1. there are no pieces occupying the space on the other side of the reflection, and

[2] Furthermore, the *sequence* of x-rotations introduced in [Cha03] represents a type of translation move (described later in Section 3.2).

2. the reflective joints are free to move 180° in the reflection direction (i.e. either clockwise or counterclockwise).

It not difficult to see when the latter requirement is satisfied (namely, when the reflection axis is exactly the axis of rotation of each of the joints). Fig. 6 shows some examples of successful reflection moves.

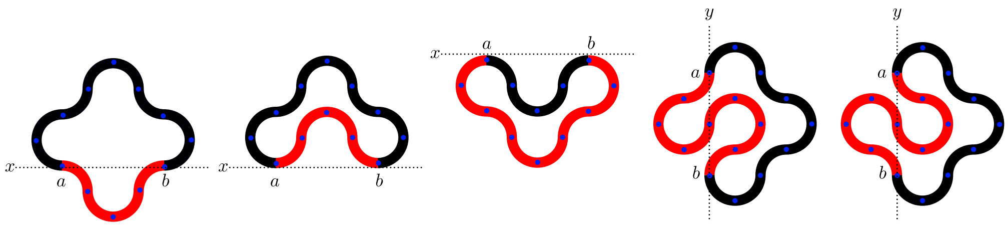

Figure 6: Reflection moves over horizontal and vertical axes where the joints labeled a and b are the reflection joints. Reflections may or may not change the orientation of the reflected links.

A reflection over the axis can change the orientation of a link. For example, Fig. 6 shows the result of reflecting a chain of links over the indicated x- or y-axis, resulting in the final configuration where all the orientations of the reflected links have changed. Although, there are examples of reflections where the orientation of the links do not change (see the middle figure in Fig. 6).

Some planar Tangle configurations may allow no reflection moves. Fig. 7 shows two instances where no reflection moves are possible.

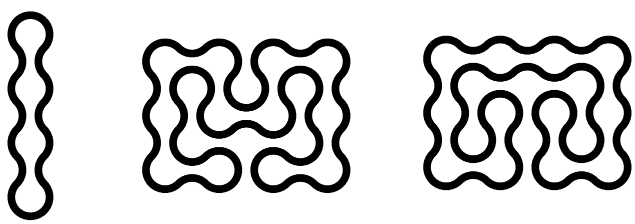

Figure 7: The first two planar Tangle configurations do not allow any reflection moves. The third configuration allows no translation moves.

3.2 Translations

Translations are asymmetric moves that are not performed across a single axis, but across a collection of parallel axes. A translation has two *translation joints* oriented in the same (vertical or horizontal) direction and four *translation links*, the two links next to each of the translation joints. When a translation move is performed, one of the two connected components of the Tangle without its translation links is picked up and translated to a different location relative to the other component by rotating the translation links. Fig. 8 shows an example of a translation move.

Translation involves the rotation of the four links connected to the translational joints. A translation move may only be made if

1. the translational links can be rotated, and

2. the translated portion may be placed in a location that do not contain other links.

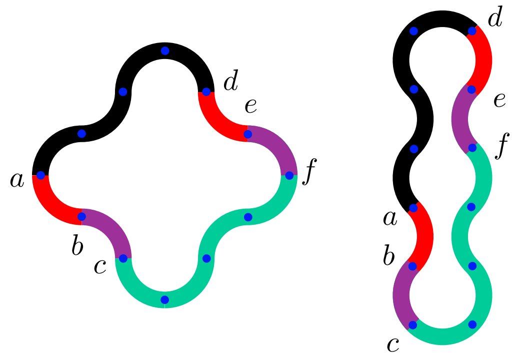

Figure 8: Example of translation move. Translation involves the rotation of all four links connected to each of the two translational joints indicated by b and e. Here the translation move rotates the links spanned by the joints a, b, c, d, e, and f.

Fig. 7 shows one example where no translation moves are allowed.

The natural question (answered in Section 4) is whether any planar Tangle configurations can reach any other by a sequence of reflection and/or translation moves. Otherwise, we call an n-Tangle *locked*, meaning that a proper subset of the planar configurations cannot reach configurations outside the set. In particular, we call a planar Tangle configuration *rigid* if it admits no such reflection or translation moves.

3.3 Tangle Moves Enumerator

The Tangle Moves Enumerator [DD15b] takes a starting planar Tangle configuration and lists all possible planar configurations that can be reached via the moves above. The enumerator performs the search in a breadth-first manner. There are $O(n^2)$ possible rotation and translation axes. For each axis, the number of possible rotational moves is 2 and the number of possible translational moves is also 2. Therefore, the number of possible new configurations resulting from moves in each level of the search is $O(n^2)$. The enumerator exhaustively searches each possible new configuration. If a configuration was already reached before, the current branch of the search is terminated. In the next section, we use this software to explore the configurations reachable from a planar Tangle configuration to determine whether it is locked or rigid.

4 Rigid and Locked Tangles

Here we illustrate two planar Tangle structures that are rigid under the moves defined in Section 3. Furthermore, we demonstrate a set of locked but not rigid configurations with $n = 308$ links. We thereby disprove both conjectures in [Tay15] and [Cha03]. Both examples can be verified by hand or with the Tangle Moves Enumerator (Section 3.3).

4.1 Rigid Structures

Fig. 9 and Fig. 10 show two symmetric examples of rigid structures along with their dual graphs. Fig. 10 shows that, even if we restrict ourselves to planar Tangle configurations where the dual graph is a path, there exist rigid configurations.

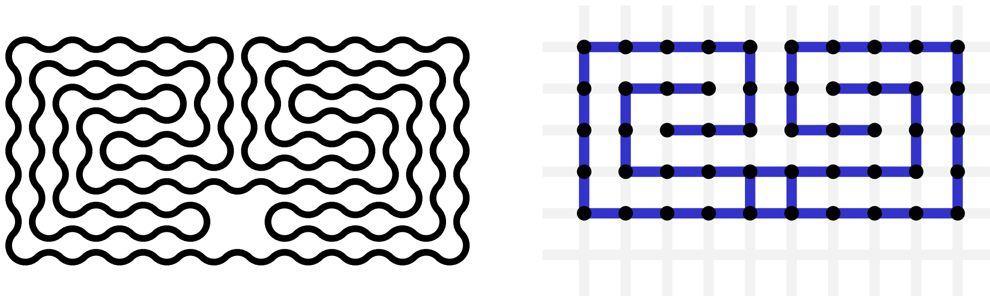

Figure 9: 4-leaf, symmetric rigid counterexample. Here, the dual graph contains 4 leaves and a cycle.

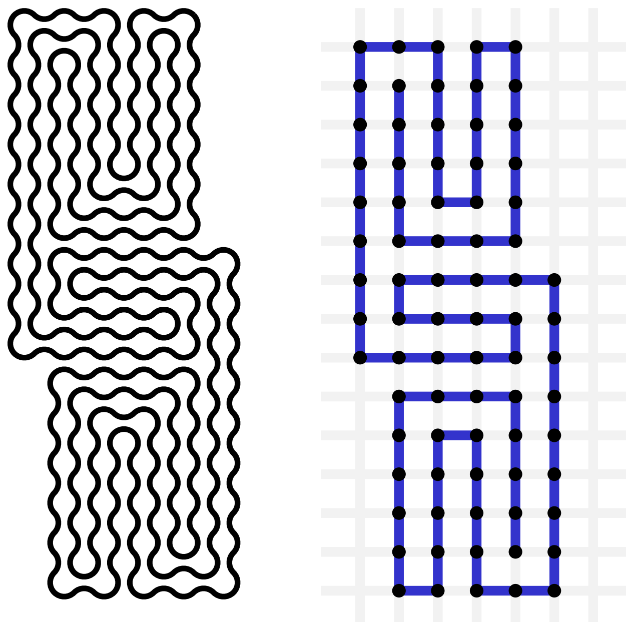

Figure 10: 2-leaf, symmetric rigid counterexample. Here, the dual graph contains 2 leaves and is a simple path.

4.2 Locked Structures

Fig. 11 shows an example of locked but not rigid Tangles: these seven planar 308-Tangles cannot reach any planar configuration outside this set. Seven is far less than the number of possible planar 308-Tangle configurations, so the set is locked.

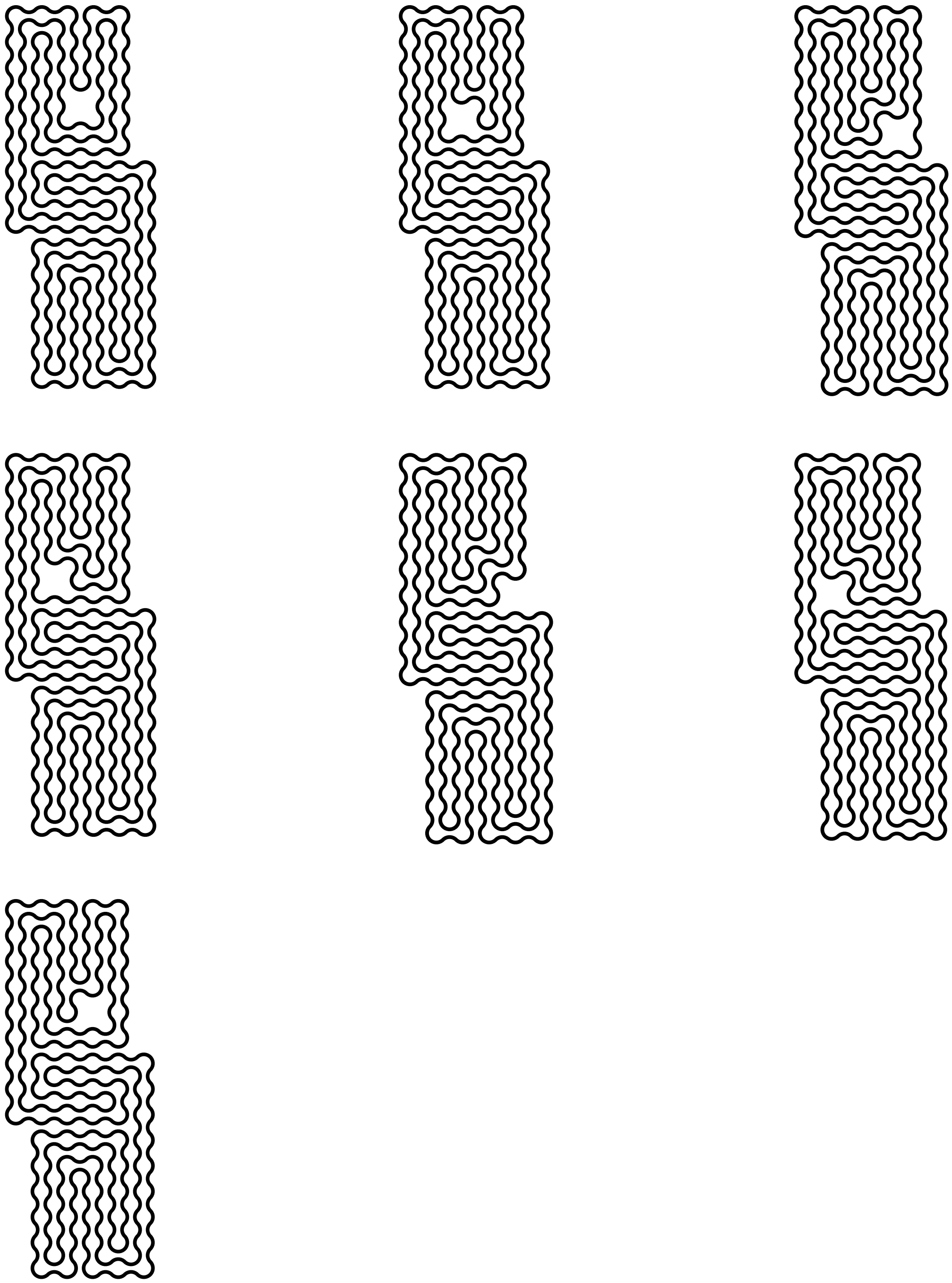

Figure 11: Locked planar 308-Tangles. Illustrated are seven planar Tangles that can reach each other but none of the other planar 308-Tangles.

5 Tangle Fonts

Mathematical typefaces offer a way to illustrate mathematical theorems and open problems, especially in computational geometry, to the general public. Previous examples include typefaces

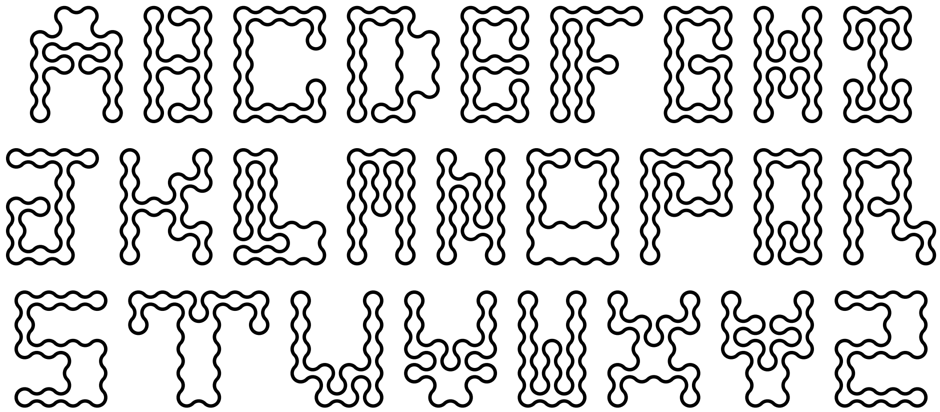

Figure 12: 52-Tanglegram typeface.

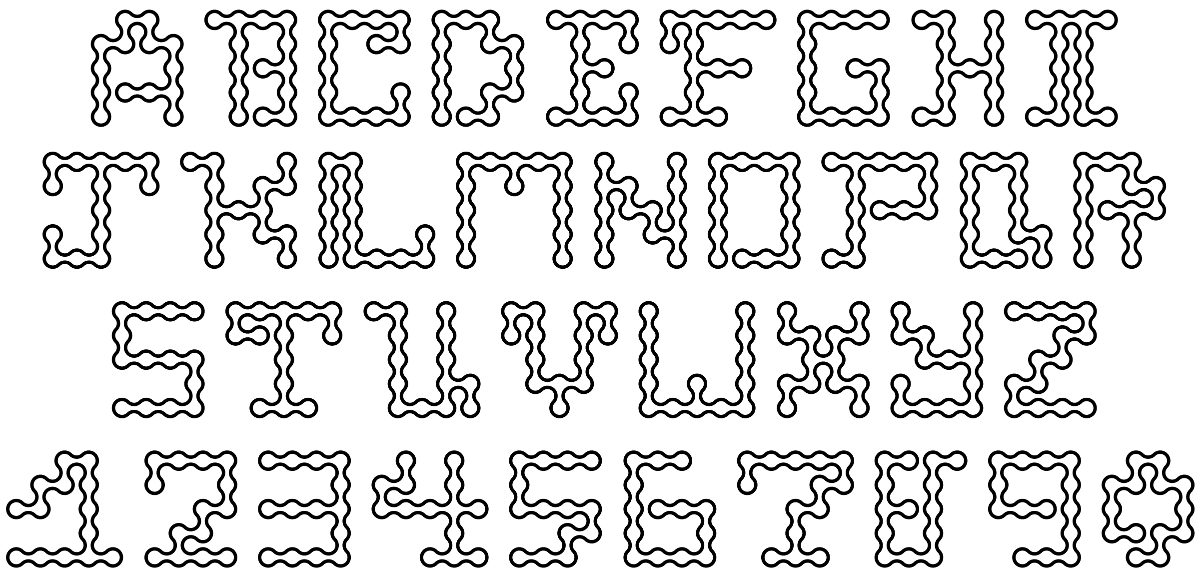

Figure 13: 56-Tanglegram typeface.

illustrating hinged dissections, origami mazes, and fixed-angle linkages; see [DD15a]. Free software lets you interact with these fonts.[3]

Here we develop two Tangle typefaces, where each letter is a planar Tangle configuration of a common length. Figures 12 and 13 show the typeface of 52- and 56-Tangles, respectively. Our software lets you write messages in these fonts.[4] We know that these configurations can reach each other by complex 3D motions without collision [ADD$^+$02]. An interesting open question is whether all the configurations in each font can reach each other via just reflection and translation moves. We conjecture the answer is "yes"; see Figure 14 for one example.

[3]http://erikdemaine.org/fonts/

[4]http://erikdemaine.org/fonts/tangle/

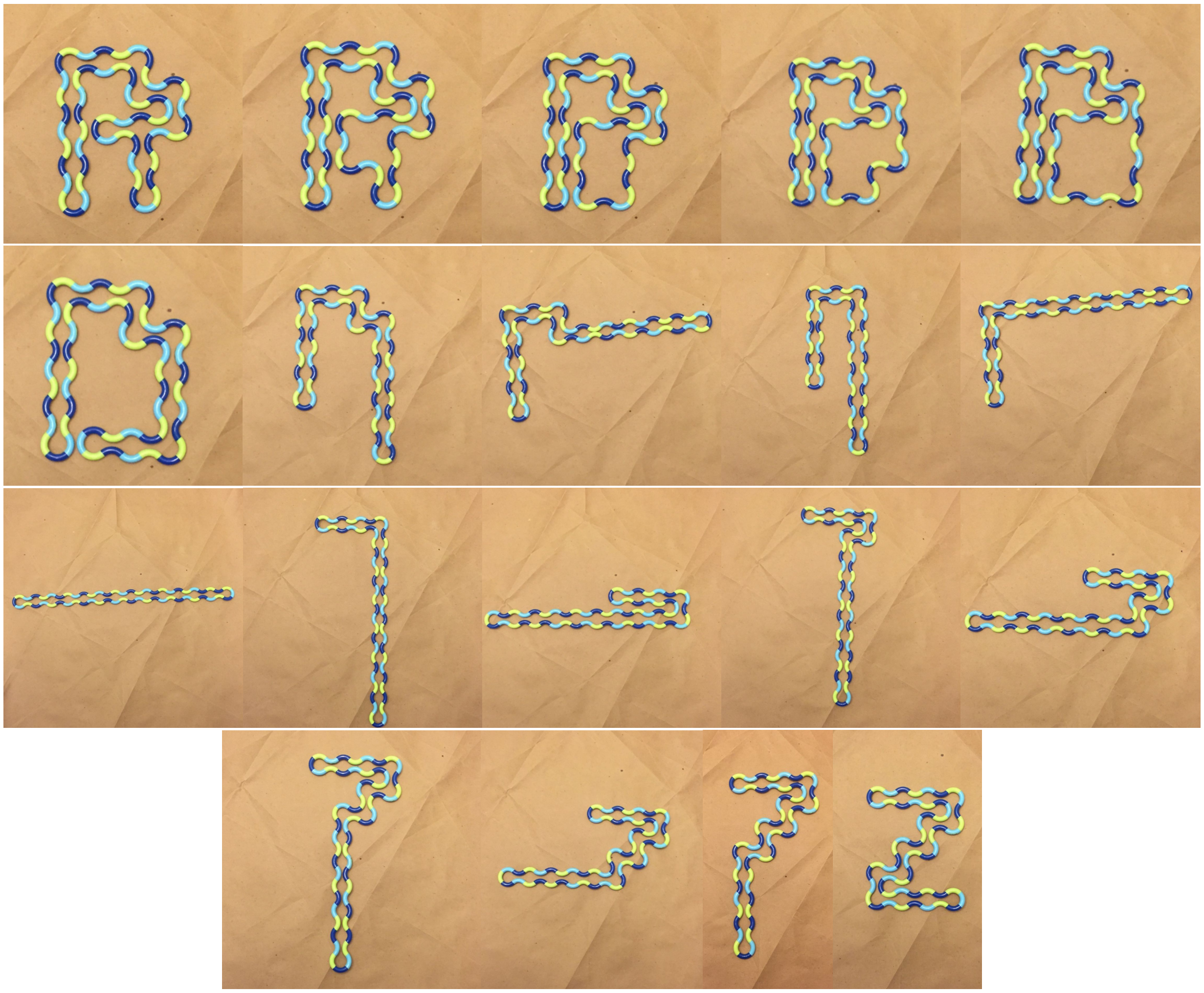

Figure 14: Transforming R into Z, in honor of Richard Zawitz who invented Tangle [Zaw85].

6 Open Questions

Since we showed that there exist planar locked and even rigid Tangle structures under our reflection and translation moves, a natural next step is to investigate the computational complexity of determining whether a structure is rigid. Another natural question is the computational complexity of determining whether two planar configurations of an n-Tangle can be transformed into each other through a sequence of valid moves. Furthermore, a natural optimization question is, given two planar n-Tangle configurations, to find the minimal set(s) of reflection and translation moves necessary to transform one into the other.

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