Algorithmic Graph Minor Theory: Improved Grid Minor Bounds and Wagner's Contraction

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Abstract

We explore three important avenues of research in algorithmic graph-minor theory, which all stem from a key min-max relation between the treewidth of a graph and its largest grid minor. This min-max relation is a keystone of the Graph Minor Theory of Robertson and Seymour, which ultimately proves Wagner's Conjecture about the structure of minor-closed graph properties.

First, we obtain the only known polynomial min-max relation for graphs that do not exclude any fixed minor, namely, map graphs and power graphs. Second, we obtain explicit (and improved) bounds on the min-max relation for an important class of graphs excluding a minor, namely, $K_{3,k}$-minor-free graphs, using new techniques that do not rely on Graph Minor Theory. These two avenues lead to faster fixed-parameter algorithms for two families of graph problems, called minor-bidimensional and contraction-bidimensional parameters, which include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, dominating set, edge dominating set, R-dominating set, connected dominating set, connected edge dominating set, connected R-dominating set, and unweighted TSP tour. Third, we disprove a variation of Wagner's Conjecture for the case of graph contractions in general graphs, and in a sense characterize which graphs satisfy the variation. This result demonstrates the limitations of a general theory of algorithms for the family of contraction-closed problems (which includes, for example, the celebrated dominating-set problem). If this conjecture had been true, we would have had an extremely powerful tool for proving the existence of efficient algorithms for any contraction-closed problem, like we do for minor-closed problems via Graph Minor Theory.

1 Introduction

Graph Minor Theory is a seminal body of work in graph theory, developed by Robertson and Seymour in a series of over 20 papers spanning the last 20 years. The original goal of this work, now achieved, was to prove Wagner's Conjecture [50], which can be stated as follows: every minor-closed graph property (preserved under taking of minors) is characterized by a finite set of forbidden minors. This theorem has a powerful algorithmic consequence: every minor-closed graph property can be decided by a polynomial-time algorithm. A keystone in the proof of these theorems, and many other theorems, is a grid-minor theorem [46]: any graph of treewidth at least some $f(r)$ is guaranteed to have the $r \times r$ grid graph as a minor. Such grid-minor theorems have also played a key role for many algorithmic applications, in particular via the bidimensionality theory (e.g., [21,

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14, 15, 12, 18, 17, 20]), including many approximation algorithms, PTASs, and fixed-parameter algorithms.

The grid-minor theorem of [46] has been extended, improved, and re-proved by Robertson, Seymour, and Thomas [55], Reed [43], and Diestel, Jensen, Gorbunov, and Thomassen [27]. The best bound known for general graphs is superexponential: every graph of treewidth more than 20^{2r^5} has an $r \times r$ grid minor [55]. This bound is usually not strong enough to derive efficient algorithms. Robertson et al. [55] conjecture that the bound on $f(r)$ can be improved to a polynomial $r^{\Theta(1)}$; the best known lower bound is $\Omega(r^2 \lg r)$. A tight linear upper bound was recently established for graphs excluding any fixed minor H: every H-minor-free graph of treewidth at least $c_H\, r$ has an $r \times r$ grid minor, for some constant c_H [18]. This bound leads to many powerful algorithmic results on H-minor-free graphs [18, 17, 20].

Three major problems remain in the literature with respect to these grid-minor theorems in particular, and algorithmic graph-minor theory in general. We address all three of these problems in this paper.

First, to what extent can we generalize algorithmic graph-minor results to graphs that do not exclude a fixed minor H? In particular, for what classes of graphs can the grid-minor theorem be improved from the general superexponential bound to a bound that would be useful for algorithms? To this end, we present polynomial grid-minor theorems for two classes of graphs that can have arbitrarily large cliques (and therefore exclude no fixed minors). One class, *map graphs*, is an important generalization of planar graphs introduced by Chen, Grigni, and Papadimitriou [11], characterized via a polynomial recognition algorithm by Thorup [59], and studied extensively in particular in the context of subexponential fixed-parameter algorithms and PTASs for specific domination problems [13, 10]. The other class, *power graphs*, e.g., fixed powers of H-minor-free graphs (or even map graphs), have been well-studied since the time of the Floyd-Warshall algorithm; see, e.g., [35, 40, 41, 44].

Second, even for H-minor-free graphs, how large is the constant c_H in the grid-minor theorem? In particular, how does it depend on H? This constant is particularly important because it is in the exponent of the running times of many algorithms. The current results (e.g., [18]) heavily depend on Graph Minor Theory, most of which lacks explicit bounds and is believed to have very large bounds.[1] For this reason, improving the constants, even for special classes of graphs, and presumably using different approaches from Graph Minors, is an important theoretical and practical challenge. To this end, we give explicit bounds for the case of $K_{3,k}$-minor-free graphs, an important class of apex-minor-free graphs (see, e.g., [5, 8, 29, 30]). Our bounds are not too small but are a vast improvement over previous bounds (in particular, much smaller than $2 \uparrow |V(H)|$). In addition, the proof techniques are interesting in their own right, for example, the path-intertwining technique used in many contexts (see, e.g., [4, 3, 25, 42]). To the best of our knowledge, this is the only grid-minor theorem with an explicit bound other than for planar graphs [55] and bounded-genus graphs [14]. Our theorem also leads to several algorithms with explicit and improved bounds on their running time.

Third, to what extent can we generalize algorithmic graph-minor results to graph contractions? Many graph optimization problems are closed (only decrease) under edge contractions, but not under edge deletions (i.e., minors). Examples include dominating set, traveling salesman, or even

[1]To quote David Johnson [36], "for any instance $G = (V, E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour's." He estimates one constant in an algorithm for testing for a fixed minor H to be roughly $2 \uparrow 2^{2^{2^{2\uparrow(2\uparrow\Theta(|V(H)|))}}}$, where $2 \uparrow n$ denotes a tower $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ involving n 2's.

diameter. Bidimensionality theory has been extended to such contraction-closed problems for the case of apex-minor-free graphs; see, e.g., [12, 14, 18, 17, 24]. The basis for this work is a modified grid-minor theorem which states that any apex-minor-free graph of treewidth at least $f(r)$ can be contracted into an "augmented" $r \times r$ grid (e.g., allowing partial triangulation of the faces). The ultimate goal of this line of research, mentioned explicitly in [16, 24], is to use this grid-contraction analog of the grid-minor theorem to develop a Graph Contraction Theory paralleling as much as possible of Graph Minor Theory. In particular, the most natural question is whether Wagner's Conjecture generalizes to contractions: is every contraction-closed graph property characterized by a finite set of excluded contractions? If this were true, it would generalize our algorithmic knowledge of minor-closed graph problems in a natural way to the vast array of contraction-closed graph problems. To this end, we unfortunately disprove this contraction version of Wagner's Conjecture, even for planar bounded-treewidth graphs. On the other hand, we prove that the conjecture holds for outerplanar graphs and triangulated planar graphs, which in some sense provides a tight characterization of graphs for which the conjecture holds.

Below we detail our results and techniques for each of these three problems.

1.1 Our Results and Techniques

Generalized grid-minor bounds. We establish polynomial relations between treewidth and grid minors for map graphs and for powers of graphs. We prove in Section 3 that any map graph of treewidth at least r^3 has an $\Omega(r) \times \Omega(r)$ grid minor. We prove in Section 4 that, for any graph class with a polynomial relation between treewidth and grid minors (such as H-minor-free graphs and map graphs), the family of kth powers of these graphs also has such a polynomial relation, where the polynomial degree is larger by just a constant, interestingly independent of k.

These results extend bidimensionality to map graphs and power graphs, improving the running times of a broad class of fixed-parameter algorithms for these graphs. See Section 5 for details on these algorithmic implications. Our results also build support for Robertson, Seymour, and Thomas's conjecture that all graphs have a polynomial relation between treewidth and grid minors [55]. Indeed, from our work, we refine the conjecture to state that all graphs of treewidth $\Omega(r^3)$ have an $\Omega(r) \times \Omega(r)$ grid minor, and that this bound is tight. The previous best treewidth-grid relations for map graphs and power graphs were given by the superexponential bound from [55].

The main technique behind these results is to use efficient min-max relations between treewidth and the size of a grid minor. In contrast, most previous work uses the seminal efficient min-max relation between treewidth and tangles or between branchwidth and tangles, proved by Robertson and Seymour [54], or the inefficient min-max relations between treewidth and grid minors. We show that grids are powerful structures that are easy to work with. By bootstrapping, we use grids and their connections to treewidth even to prove relations between grids and treewidth.

Another example of the power of this technique is a result we obtain in Section 6 as a byproduct of our study of map graphs: every bounded-genus graph has treewidth within a constant factor of the treewidth of its dual. This is the first relation of this type for bounded-genus graphs. The result generalizes a conjecture of Seymour and Thomas [56] that, for planar graphs, the treewidth is within an additive 1 of the treewidth of the dual, which has been proved in [39, 6] using a complicated approach. Such a primal-dual treewidth relation is useful, e.g., for bounding the change in treewidth when performing operations in the dual. Our proof crucially uses the connections between treewidth and grid minors, and this approach leads to a relatively clean argument. The tools we use come from bidimensionality theory and graph contractions, even though the result is not explicitly about either.

Explicit (improved) grid-minor bounds. We prove in Section 7 that the constant c_H in the linear grid-minor bound for H-minor-free graphs can be bounded by an explicit function of $|V(H)|$ when $H = K_{3,k}$ for any k: for an explicit constant c, every $K_{3,k}$-minor-free graph of treewidth at least $c^k r$ has an $r \times r$ grid minor. This bound makes explicit and substantially improves the constants in the exponents of the running time of many fixed-parameter algorithms from bidimensionality theory [14, 12, 18] for such graphs. $K_{3,k}$-minor-free graphs play an important role as part of the family of apex-minor-free graphs that is disjoint from the family of single-crossing-minor-free graphs (for which there exist a powerful decomposition in terms of planar graphs and bounded-treewidth graphs [52, 21]). Here the *family of $\mathcal{X}$-minor-free graphs* denotes the set of X-minor-free graphs for any fixed graph X in the class $\mathcal{X}$. $K_{3,k}$ is an *apex graph* in the sense that it has a vertex whose removal leaves a planar graph. For $k \geq 7$, $K_{3,k}$ is not a *single-crossing graph* in the sense of being a minor of a graph that can be drawn in the plane with at most one crossing: $K_{3,k}$ has genus at least $(k-2)/4$, but a single-crossing graph has genus at most 1 (because genus is closed under minors).

There are several structural theorems concerning $K_{3,k}$-minor-free graphs. According to Robertson and Seymour (personal communication—see [8]), $K_{3,k}$-minor-free graphs were the first step toward their core result of decomposing graphs excluding a fixed minor into graphs almost-embeddable into bounded-genus surfaces, because $K_{3,k}$-minor-free graphs can have arbitrarily large genus. Oporowski, Oxley, and Thomas [42] proved that any large 3-connected $K_{3,k}$-minor-free graph has a large wheel as a minor. Böhme, Kawarabayashi, Maharry, and Mohar [3] proved that any large 7-connected graph has a $K_{3,k}$ minor, and that the connectivity 7 is best possible. Eppstein [29, 30] proved that a subgraph P has a linear bound on the number of times it can occur in $K_{3,k}$-minor-free graphs if and only if P is 3-connected.

Our explicit linear grid-minor bound is based on an approach of Diestel et al. [27] combined with arguments in [5, 3] to find a $K_{3,k}$ minor. Using similar techniques we also give explicit bounds on treewidth for a theorem decomposing a single-crossing-minor-free graph into planar graphs and bounded-treewidth graphs [52, 21], when the single-crossing graph is $K_{3,4}$ or K_6^- (K_6 minus one edge). Both proofs must avoid Graph Minor Theory to obtain the first explicit bounds of their kind.

Contraction version of Wagner's Conjecture. Wagner's Conjecture, proved in [50], is a powerful and very general tool for establishing the existence of polynomial-time algorithms; see, e.g., [31]. Combining this theorem with the $O(n^3)$-time algorithm for testing whether a graph has a fixed minor H [49], every minor-closed property has an $O(n^3)$-time decision algorithm which tests for the finite set of excluded minors. Although these results are existential, because the finite set of excluded minors is not known for many minor-closed properties, polynomial-time algorithms can often be constructed [19].

A natural goal is to try to generalize these results even further, to handle all contraction-closed properties, which include the decision versions of many important graph optimization problems such as dominating set and traveling salesman, as well as combinatorial properties such as diameter. Unfortunately, we show in Section 8 that the contraction version of Wagner's Conjecture is not true: there is a contraction-closed property that has no complete finite set of excluded contractions. Our counterexample has an infinite set of excluded contractions all of which are planar bounded-treewidth graphs. On the other hand, we show that the contraction version of Wagner's Conjecture holds for trees, triangulated planar graphs, and 2-connected outerplanar graphs: any contraction-closed property characterized by an infinite set of such graphs as contractions can be characterized by a finite set of such graphs as contractions. Thus we nearly characterize the set of graphs for which the contraction version of Wagner Conjecture's is true. The proof for outerplanar graphs is the most complicated, and uses Higman's theorem on well-quasi-ordering [34].

The reader is referred to the full version of this paper (available from the first author's website) for the proofs. See also [16] for relevant definitions.

2 Definitions and Preliminaries

Treewidth. The notion of treewidth was introduced by Robertson and Seymour [53]. To define this notion, first we consider a representation of a graph as a tree, called a tree decomposition. A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$ such that the following two properties hold:

(W1) $\bigcup_{t\in V(T)} Y_t = V(G)$, and every edge of G has both endpoints in some Y_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T, Y) is $\max_{t\in V(T)} |Y_t| - 1$. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G.

Oporowski et al. [42] show that, if a graph G has a tree decomposition of width at most w, then G has a tree decomposition of width at most w that further satisfies the following properties:

(W3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.

(W4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.

(W5) If $t_0 \in V(T)$ and B is a component of $T - t_0$, then $\bigcup_{t\in V(B)} Y_t \setminus Y_{t_0} \neq \emptyset$.

Minors and contractions. Given an edge $e = \{v, w\}$ in a graph G, the *contraction* of e in G is the result of identifying vertices v and w in G and removing the self-loop and any duplicate edges. A graph H obtained by a sequence of such edge contractions starting from G is said to be a *contraction* of G. A graph H is a *minor* of G if H is a subgraph of some contraction of G. A graph class $\mathcal{C}$ is *minor-closed* if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor-free* if $H \notin \mathcal{C}$. More generally, we use the term "H-minor-free" to refer to any minor-closed graph class that excludes some fixed graph H.

Grid minors. We use the following important connections between treewidth and the size of the largest grid minor. The $r \times r$ *grid* is the planar graph with r^2 vertices arranged on a square grid and with edges connecting horizontally and vertically adjacent vertices. First we state the connection for planar graphs:

Theorem 1 [55] *Every planar graph of treewidth w has an $\Omega(w + 1) \times \Omega(w + 1)$ grid graph as a minor.*[2]

The more general connection for H-minor-free graphs has been obtained recently:

Theorem 2 [18] *For any fixed graph H, every H-minor-free graph of treewidth w has an $\Omega(w + 1) \times \Omega(w + 1)$ grid graph as a minor.*

[2]We require bounds involving asymptotic notation O, Ω, and Θ to hold for all values of the parameters, in particular, w. Thus, $\Omega(w + 1)$ has a different meaning from $\Omega(w)$ when $w = 0$. In this theorem, when the treewidth is 0, i.e., the graph has no edges, there is still a 1×1 grid.

Walls. An *r-wall* is a graph isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i, j) \mid 1 \le i \le r, 1 \le j \le r\}$ in which two vertices (i, j) and (i', j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j - 1, j + 1\}$.

(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

We can define an $a \times b$ wall in a similar way. It is easy to see that, if G has an $a \times b$-wall, then it has an $a \times b$ grid minor, and conversely, if G has an $a \times b$ grid minor, then it has an $a/2 \times b$ wall. Let us recall that the $a \times b$ grid is the Cartesian product of paths $P_a \times P_b$. Figure 1 shows the 4×5 grid and the 8×5 wall.

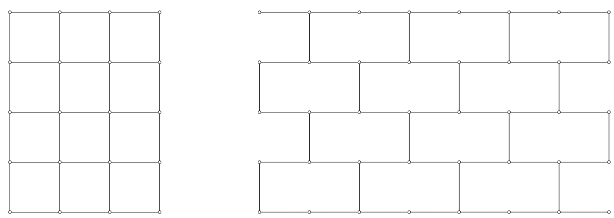

Figure 1: The (4×5)-grid and the (8×5)-wall

Embeddings. A *2-cell embedding* of a graph G in a surface Σ (two-dimensional manifold) is a drawing of the vertices as points in Σ and the edges as curves in Σ such that no two points coincide, two curves intersect only at shared endpoints, and every face (region) bounded by edges is an open disk. We define the *Euler genus* or simply *genus* of a surface Σ to be the "nonorientable genus" or "crosscap number" for nonorientable surfaces Σ, and twice the "orientable genus" or "handle number" for orientable surfaces Σ. The *(Euler) genus* of a graph G is the minimum genus of a surface in which G can be 2-cell embedded. A graph has *bounded genus* if its genus is $O(1)$.

A *planar embedding* is a 2-cell embedding into the plane (topological sphere). An *embedded planar graph* is a graph together with a planar embedding.

Planar up to 3-separations. Suppose G_0 can be written as $G_1 \cup G_2 \cup \ldots,$, where $G_1 \cap G_i = \{v_1, \ldots, v_t\} \subset V(G_0)$ for $i = 1, 2, \ldots,$ $1 \le t \le 3$. Furthermore $V(G_i) \setminus V(G_1) \neq \emptyset$ if $t \le 2$ and $|V(G_i) \setminus V(G_1)| \ge 2$ if $t = 3$ for $i = 1, 2, \ldots$. Then we replace G_0 by the graph G' obtained from G_1 by adding all edges v_iv_j $(1 \le i < j \le t)$ that are not already in G_1. If G' is planar, then we say that G_0 is *planar, up to 3-separations.* Roughly, one can think that G_1 is completely embedded into the plane, while G_i is not, but is attached to a cuff of G_1 with $|G_1 \cup G_i| \le 3$ for $i = 1, \ldots$.

Map graphs and power graphs. We consider two classes of graphs that can have arbitrarily large cliques and therefore do not exclude any fixed minor. Given an embedded planar graph and a partition of its faces into *nations* or *lakes*, the associated *map graph* has a vertex for each nation and an edge between two vertices corresponding to nations (faces) that share a vertex. This modified definition of the dual graph was introduced by Chen, Grigni, and Papadimitriou [11] as a generalization of planar graphs that can have arbitrarily large cliques. Later Thorup [59] gave a polynomial-time algorithm for recognizing map graphs and reconstructing the planar graph and the partition.

We can view the class of map graphs as a special case of taking powers of a family of graphs. The *kth power* G^k of a graph G is the graph on the same vertex set $V(G)$ with edges connecting two vertices in G^k precisely if the distance between these vertices in G is at most k. For a bipartite graph G with bipartition $V(G) = U \cup W$, the *half-square* $G^2[U]$ is the graph on one side U of the partition, with two vertices adjacent in $G^2[U]$ precisely if the distance between these vertices in G is 2. A graph is a map graph if and only if it is the half-square of some planar bipartite graph [11]. In fact, this translation between map graphs and half-squares is constructive and takes polynomial time.

Duals and maps. More formally, we define a map graph and related notions in terms of an embedded planar graph G and a partition of faces into a collection $N(G)$ of *nations* and a collection $L(G)$ of *lakes*. Thus, $N(G) \cup L(G)$ is the set of faces of G.

We define the *(modified) dual* $D = D(G)$ of G in terms of only the nations of G. The graph D has a vertex for every nation of G, and two vertices are adjacent in D if the corresponding nations of G share an edge.

The *map graph* $M = M(G)$ of G has a vertex for every nation of G, and two vertices are adjacent in $M(G)$ if the corresponding nations of G share a vertex. The dual graph $D(G)$ is a subgraph of the map graph $M(G)$.

Canonical map graphs. We canonicalize G in the following ways that preserve the map graph $M(G)$. First, we remove any vertex of G incident only to lakes, because it and its incident edges do not contribute to the map graph $M(G)$. Second, for any edge of G whose two incident faces are both lakes (possibly the same lake), we delete the edge and merge the corresponding lakes, because again this will not change the map graph $M(G)$.

Third, we modify G to ensure that every vertex is incident to at most one lake, and incident to such a lake at most once. Consider a vertex v that violates this property, and suppose there is an incident lake between edges $\{v, w_i\}$ and $\{v, w'_i\}$ for $i = 1, 2, \ldots, l$. We split v into $l + 1$ vertices $v, v_1, v_2, \ldots, v_l$, with v_i placed near v in the wedge $w_i, v, w'_i\}$. We connect these $l + 1$ vertices in a star, with an edge between v and v_i for $i = 1, 2, \ldots, l$. Edges $\{v, w_i\}$ and $\{v, w'_i\}$ reroute to be $\{v_i, w_i\}$ and $\{v_i, w'_i\}$, and all other edges incident to v remain as they were. as in the second canonicalization. This modification preserves the map graph $M(G)$ and results in no lakes touching at v.

Finally, we assume that the map graph $M(G)$ is connected, because we can always consider each connected component separately.

Radial graphs. The *radial graph* $R = R(G)$ has a vertex for every vertex of G and for every nation of G, and we label them the same: $V(R) = V(G) \cup N(G)$. $R(G)$ is bipartite with this bipartition. Two vertices $v \in V(G)$ and $f \in N(G)$ are adjacent in $R(G)$ if their corresponding vertex v and nation f are incident.

We also consider the union graph $R \cup D$. $R \cup D$ has the same vertex set as the radial graph R, which is a superset of the vertex set of the dual graph D. The edges in $R \cup D$ consist of all edges in R and all edges in D.

We also define the *radial graph* $R = R(G)$ for a graph G 2-cell embedded in an arbitrary surface Σ. In this case, we do not allow lakes, and consider every face to be a nation. Otherwise, the definition is the same.

3 Treewidth-Grid Relation for Map Graphs

In this section we prove a polynomial relation between the treewidth of a map graph and the size of the largest grid minor. The main idea is to relate the treewidth of the map graph $M(G)$, the treewidth of the radial graph $R(G)$, the treewidth of the dual graph $D(G)$, and the treewidth of the union graph $R(G) \cup D(G)$.

Lemma 3 *The treewidth of the union $R \cup D$ of the radial graph R and the dual graph D, plus* 1, *is within a constant factor of the treewidth of the dual graph D, plus* 1.

Proof: First, $\text{tw}(D) + 1 \leq \text{tw}(R \cup D) + 1$ because D is a subgraph of $R \cup D$.

The rest of the proof establishes that $\text{tw}(D) + 1 = \Omega(\text{tw}(R \cup D) + 1)$. Because both graphs are planar, we know by Theorem 1 that 1 plus the treewidth of either graph is within a constant factor of the dimension of the largest grid minor. Thus it suffices to show that we can convert a given $k \times k$ grid minor K of $R \cup D$ into an $\Omega(k) \times \Omega(k)$ grid minor of D.

Consider the sequence of edge contractions and removals that bring $R \cup D$ to the grid K. Discard all edge deletions from this sequence, but remove any loops and duplicate copies of edges that arise from contractions. The resulting graph K' remains planar and has the same vertices as K, and therefore K' is a partially triangulated $k \times k$ grid, in the sense that each face of the $k \times k$ grid can have a noncrossing set of additional edges. (All bounded faces of the grid have 4 vertices and so at most one additional edge.)

We label each vertex v in K' with the set of vertices from $R \cup D$ that contracted to form v. We call v *facial* if at least one of these vertices is a vertex of the dual graph D. Otherwise, v is *nonfacial*. No two nonfacial vertices can be adjacent in K', because no two vertices in G are adjacent in $R \cup D$.

Assign coordinates (x, y), $0 \leq x, y < k$, to each vertex v in K'. We assume without loss of generality that k is divisible by 6 (decreasing k by at most 5 if necessary). For each i, j with $1 \leq i, j \leq k/6 - 1$, either vertex $(6i + 1, 6j + 1)$ or vertex $(6i + 2, 6j + 1)$ is facial, because these two vertices are adjacent in K'. Let $v_{i,j}$ denote a facial vertex among this pair. Let $\hat{v}_{i,j}$ denote a vertex of the dual graph D in the label of $v_{i,j}$ (which exists by the definition of facial).

For any i, j with $1 \leq i \leq k/6 - 1$ and $1 \leq j \leq k/6 - 2$, we claim that there is a simple path between $\hat{v}_{i,j}$ and $\hat{v}_{i,j+1}$ in D using only vertices in D that appear in the labels of vertices in R' with coordinates in the rectangle $(6i\,..\,6i+3, 6j\,..\,6(j+1)+3)$. We start with a shortest path $P_{K'}$ between $v_{i,j}$ and $v_{i,j+1}$ in K', which is simple and remains in the subrectangle $(6i + 1\,..\,6i + 2, 6j + 1\,..\,6(j + 1) + 2)$. We convert $P_{K'}$ into a simple path $P_{R \cup D}$ between $\hat{v}_{i,j}$ and $\hat{v}_{i,j+1}$ in $R \cup D$ using only the vertices in $R \cup D$ that appear in the labels of the vertices in K' along $P_{K'}$. Here we use that the subgraph of $R \cup D$ induced by the label set of a vertex in K' is connected, because that vertex in K' was formed by contracting edges in this subgraph. For each edge in the path $P_{K'}$, we pick an edge in $R \cup D$ that forms it as a result of the contractions; then we connect together the endpoints of these edges, and connect the first and last edges to $\hat{v}_{i,j}$ and $\hat{v}_{i,j+1}$ respectively, by finding shortest paths within the subgraphs of $R \cup D$ induced by label sets. Finally we convert this path $P_{R \cup D}$ into a simple path P_D in D with the desired properties. The vertices along the path $P_{R \cup D}$ divide into two classes: those in D (corresponding to nations of G) and those in G (corresponding to vertices of G). Among the subsequence of vertices along the path $P_{R \cup D}$, restricted to vertices in D, we claim that every two consecutive vertices v, w can be connected using only vertices in D that appear in the labels of vertices in the desired rectangle. If v and w are consecutive along the path $P_{R \cup D}$, then they are adjacent in D and we are done. Otherwise, v and w are separated in the path $P_{R \cup D}$ by one vertex u of G (because no two vertices of G are adjacent in $R \cup D$). In G, this situation corresponds to two nations v and w that share the vertex u. Because of our canonicalization, u is incident to at most one lake, at most once, and therefore there is a sequence of nations $v = f_1, f_2, \ldots, f_j = w$ in clockwise or counterclockwise order around u. Thus in D we obtain a path $v = f_1, f_2, \ldots, f_j = w$. Each f_i is incident to u and therefore has distance 1 from u in $R \cup D$. Because the contractions that formed K' from $R \cup D$ only decrease distances, the vertices of K' with labels including f_i and u have distance at most 1 in K'. Therefore each f_i is in a label of a vertex within the thickened rectangle $(6i..6i+3, 6j..6(j+1)+3)$. If the path is not simple, we can take the shortest path between its endpoints in the subgraph induced by the vertices of the path, and obtain a simple path.

Symmetrically, for any i, j with $1 \le i \le k/6 - 2$ and $1 \le j \le k/6 - 1$, we obtain that there is a simple path between $\hat{v}_{i,j}$ and $\hat{v}_{i+1,j}$ in D using only vertices in D that appear in the labels of vertices in R' with coordinates in the rectangle $(6i..6(i+1)+3, 6j..6j+3)$.

We construct a grid minor K'' of D as follows. We start with the union, over all i, j, of the simple path between $\hat{v}_{i,j}$ and $\hat{v}_{i,j+1}$ in D and the simple path between $\hat{v}_{i,j}$ and $\hat{v}_{i+1,j}$ in D. (In other words, we delete all vertices not belonging to one of these paths.) Then we contract every vertex in this union that is not one of the $\hat{v}_{i,j}$'s toward its "nearest" $\hat{v}_{i,j}$. More precisely, for each path between $\hat{v}_{i,j}$ and $\hat{v}_{i,j+1}$, we cut the path at the first edge that crosses from row $6i+4$ to row $6i+5$; then we contract all vertices in the path before the cut into vertex $\hat{v}_{i,j}$, and we contract all vertices in the path after the cut into vertex $\hat{v}_{i,j+1}$. Similarly we cut each path between $\hat{v}_{i,j}$ and $\hat{v}_{i+1,j}$ at the first edge that crosses from column $6i+4$ to column $6i+5$, and contract accordingly. Because of the rectangular bounds on each path, the rectangle $(6i..6i+3, 6j+4..6j+5)$ is intersected by a unique path, the one from $\hat{v}_{i,j}$ to $\hat{v}_{i,j+1}$, and the rectangle $(6i+4..6i+5, 6j..6j+3)$ is intersected by a unique path, the one from $\hat{v}_{i,j}$ to $\hat{v}_{i+1,j}$. Hence our contraction process does not merge paths that were not originally incident (at one of the $\hat{v}_{i,j}$'s). Also, because each path is simple and strays by distance at most 1 from the original shortest path in the grid K', the vertices before the cut are disjoint from the vertices after the cut in the path. Therefore, each vertex on a path contracts to a unique vertex $\hat{v}_{i,j}$, and each path contracts to a single edge between $\hat{v}_{i,j}$ and either $\hat{v}_{i,j+1}$ or $\hat{v}_{i+1,j}$. Thus we obtain a $(k/6-1) \times (k/6-1)$ grid minor K'' of D. □

Lemma 4 *The treewidth of the map graph M is at most the product of the maximum degree of a vertex in G and* $\mathrm{tw}(R)+1$*, one more than the treewidth of the radial graph R.*

Proof: Suppose we have a tree decomposition (T, χ) of the radial graph R of width w. We modify this tree decomposition into another tree decomposition (T, χ') by replacing each occurrence of a vertex $v \in V(G)$ in a bag $\mathcal{B}$ of χ with all nations incident to v. Thus, bags in χ' consist only of nations.

We claim (T, χ') is a tree decomposition of M. First, observe that every vertex of the map graph M appears in some bag $\mathcal{B}$ of χ', because nations are vertices in the radial graph as well, so every nation appears in a bag of χ.

Second, we claim that every vertex of the map graph M appears in a connected subtree of bags in (T, χ'). A nation f appears in a bag $\mathcal{B}'$ of χ' if either it appears in the corresponding bag $\mathcal{B}$ of χ or one of its vertices appears in corresponding bag $\mathcal{B}$ of χ. The set of bags in χ containing the nation f forms a connected subtree of T, and the set of bags in χ containing any vertex v of f forms a connected subtree of T. These two subtrees, for any choice of v, overlap in at least one node of T because v and f are adjacent in the radial graph R, and thus this edge (v, f) appeared in some bag of χ. Therefore the union of the subtree of T induced by f and all vertices v of f is connected. This union is precisely the set of nodes in T whose bags in χ' contain f.

Third, we claim that every edge of the map graph M appears in some bag of χ'. An edge arises in M when two nations f_1, f_2 share a vertex v in G. This vertex v appears in some bag $\mathcal{B}$ of χ, and in constructing χ' we replaced v with nations f_1, f_2, and possibly other nations. Therefore f_1 and f_2 appear in the corresponding bag $\mathcal{B}'$ of χ'.

Finally we claim that the size of any bag $\mathcal{B}'$ in χ' is at most the maximum degree Δ of a vertex in G times the size of the corresponding bag $\mathcal{B}$ in χ. This claim follows from the construction because each vertex is replaced by at most Δ nations in the transformation from $\mathcal{B}$ to $\mathcal{B}'$. The size of each original bag $\mathcal{B}$ in χ is at most one more than the treewidth of R. Therefore the maximum bag size in χ' is at most $\Delta(\mathrm{tw}(R) + 1)$, and the treewidth of M is at most one less than this maximum bag size. □

Theorem 5 *If the treewidth of the map graph M is r^3, then it has an $\Omega(r) \times \Omega(r)$ grid as a minor.*

Proof: By Lemma 4, $\mathrm{tw}(M) = O(\Delta \cdot \mathrm{tw}(R))$. Because R is a subgraph of $R \cup D$, $\mathrm{tw}(M) = O(\Delta \cdot \mathrm{tw}(R \cup D))$. By Lemma 3, $\mathrm{tw}(M) = O(\Delta \cdot (\mathrm{tw}(D) + 1))$. Thus, if $\mathrm{tw}(M) = \Omega(r^3)$, then either $\mathrm{tw}(D) = \Omega(r)$ or $\Delta = \Omega(r^2)$. In the former case, D is a planar subgraph of M and therefore D and M have an $\Omega(r) \times \Omega(r)$ grid as a minor by Theorem 1. In the latter case, M has a $K_\Delta = K_{\Omega(r^2)}$ clique as a subgraph, and therefore has an $\Omega(r) \times \Omega(r)$ grid as minor. □

Next we show that this theorem cannot be improved from $\Omega(r^3)$ to anything $o(r^2)$:

Proposition 6 *There are map graphs whose treewidth is $r^2 - 1$ and whose largest grid minor is $r \times r$.*

Proof: Let G be an embedded wheel graph with r^2 spokes. We set all r^2 bounded faces to be nations and the exterior face to be a lake. Then the dual graph D is a cycle, and the map graph M is the clique K_{r^2}. Therefore M has treewidth $r^2 - 1$, yet its largest grid minor is $r \times r$. □

Robertson, Seymour, and Thomas [55] prove a stronger lower bound of $\Theta(r^2 \lg r)$ but only for the case of general graphs.

4 Treewidth-Grid Relation for Power Graphs

In this section we prove a polynomial relation between the treewidth of a power graph and the size of the largest grid minor. The technique here is quite different, analyzing how a radius-r neighborhood in the graph can be covered by radius-$(r/2)$ neighborhoods—a kind of "sphere packing" argument.

Theorem 7 *Suppose that, if graph G has treewidth at least cr^α for constants $c, \alpha > 0$, then G has an $r \times r$ grid minor. For any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least $cr^{\alpha+4}$ (respectively, $cr^{\alpha+6}$), then it has an $r \times r$ grid minor.*

Proof: Let $\Delta(G^k)$ denote the maximum degree of any vertex in G^k, that is, the maximum size of the k-neighborhood of a vertex in G. First we claim that $\mathrm{tw}(G^k) \leq \Delta(G^k)\,\mathrm{tw}(G)$. Consider a tree decomposition (T, χ) of G. Replace each occurrence of vertex v in χ_x with the entire radius-k neighborhood of v in G. Thus we expand the maximum bag size by a factor of at most $\Delta(G^k)$, and the width of the resulting (T, χ') is at most $\Delta(G^k)(\mathrm{tw}(G) + 1)$. We claim that (T, χ') is a tree decomposition of G^k. First, if two vertices v and w are adjacent in G^k, i.e., within distance k in G, then by construction they are in a common bag in (T, χ'), indeed any bag that originally contained either v or w. Second, we claim that the set of bags containing a vertex v is a connected subtree of T. In other words, we claim that any two vertices u and w that are within distance k of v, which give rise to occurrences of v in χ', can be connected via a path in T along which the bags always contain v. Concatenate the shortest path $u = v_0, v_1, \ldots, v_j = v$ from u to v in G and the shortest path $v = v_j, v_{j+1}, \ldots, v_l = w$ from v to w in G, both of which use vertices v_i always within distance k of v. Now construct the desired path in T by visiting, for each i in turn, the subtree of bags in χ containing occurrences of v_i, whose corresponding bags in χ' contain occurrences of v. Here we use that the bags in χ containing occurrences of v_i form a connected subtree of T, and that this subtree for v_i and this subtree for v_{i+1} share a node because v_i is adjacent to v_{i+1}.

If $\mathrm{tw}(G^k) \geq cr^{\alpha+4}$, then either $\Delta(G^k) \geq r^4$ or $\mathrm{tw}(G) \geq cr^{\alpha}$. In the latter case, we obtain by supposition that G has an $r \times r$ grid minor and thus so does the supergraph G^k. Therefore we concentrate on the former case when $\Delta(G^k) \geq r^4$. Let v be the vertex in G whose k-neighborhood N_k has maximum size, $\Delta(G^k)$. There are two cases depending on whether k is even or odd.

The simpler case is when k is even. If the $(k/2)$-neighborhood $N_{k/2}$ of v in G has size at least r^2, then in G^k we obtain a clique K_{r^2} on those vertices, so we obtain an $r \times r$ grid minor. Otherwise, label each vertex in the k-neighborhood N_k with the nearest vertex in the $(k/2)$-neighborhood $N_{k/2}$. If any vertex in the $(k/2)$-neighborhood $N_{k/2}$ is assigned as the label to at least r^2 vertices in N_k, then again we obtain a K_{r^2} clique subgraph in G^k and thus an $r \times r$ grid minor. Otherwise, the k-neighborhood N_k has size strictly less than $r^2 \cdot r^2 = r^4$, contradicting that $|N_k| = \Delta(G^k) \geq r^4$.

The case when k is odd is similar. As before, if the $\lfloor k/2 \rfloor$-neighborhood $N_{\lfloor k/2 \rfloor}$ of v in G has size at least r^2, then in G^k we obtain a clique K_{r^2} and thus an $r \times r$ grid minor. Otherwise, label each vertex in the $(k-1)$-neighborhood N_{k-1} of v with the nearest vertex in the $\lfloor k/2 \rfloor$-neighborhood $N_{\lfloor k/2 \rfloor}$. If any vertex in the $\lfloor k/2 \rfloor$-neighborhood $N_{\lfloor k/2 \rfloor}$ is assigned as the label to at least r^2 vertices in N_{k-1}, then again we obtain a K_{r^2} clique and an $r \times r$ grid. Otherwise, $|N_{k-1}| < r^4$. Finally label each vertex in N_k with the nearest vertex in N_{k-1}. If any vertex in N_{k-1} is assigned as the label to at least r^2 vertices in N_k, then again we obtain a K_{r^2} clique and an $r \times r$ grid. Otherwise, $|N_k| < r^4 \cdot r^2 = r^6$, contradicting that $|N_k| = \Delta(G^k) \geq r^6$. □

We have the following immediate consequence of Theorems 2, 5, and 7:

Corollary 8 *For any H-minor-free graph G, and for any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least r^5 (respectively, r^7), then it has an $\Omega(r) \times \Omega(r)$ grid minor. For any map graph G, and for any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least r^7 (respectively, r^9), then it has an $\Omega(r) \times \Omega(r)$ grid minor.*

5 Treewidth-Grid Relations: Algorithmic and Combinatorial Applications

Our treewidth-grid relations have several useful consequences with respect to fixed-parameter algorithms, minor-bidimensionality, and parameter-treewidth bounds.

A *fixed-parameter algorithm* is an algorithm for computing a parameter $P(G)$ of a graph G whose running time is $h(P(G))\, n^{O(1)}$ for some function h. A typical function h for many fixed-parameter algorithms is $h(k) = 2^{O(k)}$. A celebrated example of a fixed-parameter-tractable problem is vertex cover, asking whether an input graph has at most k vertices that are incident to all its edges, which admits a solution as fast as $O(kn + 1.285^k)$ [9]. For more results about fixed-parameter tractability and intractability, see the book of Downey and Fellows [28].

A major recent approach for obtaining efficient fixed-parameter algorithms is through "parameter-treewidth bounds", a notion at the heart of bidimensionality. A *parameter-treewidth bound* is an upper bound $f(k)$ on the treewidth of a graph with parameter value k. Typically, $f(k)$ is polynomial in k. Parameter-treewidth bounds have been established for many parameters; see, e.g., [1, 37, 32, 2, 7, 38, 33, 13, 21, 22, 23, 12, 15, 14]. Essentially all of these bounds can be obtained from the general theory of bidimensional parameters (see, e.g., [16]). Thus bidimensionality is the most powerful method so far for establishing parameter-treewidth bounds, encompassing all such previous results for H-minor-free graphs. However, all of these results are limited to graphs that exclude a fixed minor.

A parameter is *minor-bidimensional* if it is at least $g(r)$ in the $r \times r$ grid graph and if the parameter does not increase when taking minors. Examples of minor-bidimensional parameters include the number of vertices and the size of various structures, e.g., feedback vertex set, vertex cover, minimum maximal matching, face cover, and a series of vertex-removal parameters. Tight parameter-treewidth bounds have been established for all minor-bidimensional parameters in H-minor-free graphs for any fixed graph H [18, 12, 14].

Our results provide polynomial parameter-treewidth bounds for all minor-bidimensional parameters in map graphs and power graphs:

Theorem 9 *For any minor-bidimensional parameter P which is at least $g(r)$ in the $r \times r$ grid, every map graph G has treewidth $\mathrm{tw}(G) = O([g^{-1}(P(G))]^3)$. More generally suppose that, if graph G has treewidth at least cr^α for constants $c, \alpha > 0$, then G has an $r \times r$ grid minor. Then, for any even (respectively, odd) integer $k \geq 1$, G^k has treewidth $\mathrm{tw}(G) = O([g^{-1}(P(G))]^{\alpha+4})$ (respectively, $\mathrm{tw}(G) = O([g^{-1}(P(G))]^{\alpha+6})$). In particular, for H-minor-free graphs G, and for any even (respectively, odd) integer $k \geq 1$, G^k has treewidth $\mathrm{tw}(G) = O([g^{-1}(P(G))]^5)$ (respectively, $\mathrm{tw}(G) = O([g^{-1}(P(G))]^7)$).*

This result naturally leads to a collection of fixed-parameter algorithms, using commonly available algorithms for graphs of bounded treewidth:

Corollary 10 *Consider a parameter P that can be computed on a graph G in $h(w)\, n^{O(1)}$ time given a tree decomposition of G of width at most w. If P is minor-bidimensional and at least $g(r)$ in the $r \times r$ grid, then there is an algorithm computing P on any map graph or power graph G with running time $[h(O(g^{-1}(k))^\beta) + 2^{O([g^{-1}(k)]^\beta)}]\, n^{O(1)}$, where β is the degree of $O(g^{-1}(P(G))$ in the polynomial treewidth bound from Theorem 9. In particular, if $h(w) = 2^{O(w)}$ and $g(k) = \Omega(k^2)$, then the running time is $2^{O(k^{\beta/2})} n^{O(1)}$.*

The proofs of these consequences follow directly from combining [12] with Theorems 5 and 7 below.

In contrast, the best previous results for this general family of problems in these graph families have running times $[h(2^{O([g^{-1}(k)]^5)}) + 2^{2^{O([g^{-1}(k)]^5)}}]\, n^{O(1)}$ [12, 19].

6 Primal-Dual Treewidth Relation for Bounded-Genus Graphs

Robertson and Seymour [48, 56] proved that the branchwidth of a planar graph is equal to the branchwidth of its dual, and conjectured that the treewidth of a planar graph is within an additive 1 of the treewidth of its dual. The latter conjecture was apparently proved in [39, 6], though the proof is complicated. Here we prove that the treewidth (and hence the branchwidth) of any graph 2-cell embedded in a bounded-genus surface is within a constant factor of the treewidth of its dual. Thus the result applies more generally, though the connection is slightly weaker (constant factor instead of additive constant).

We crucially use the connection between treewidth and grids to obtain a relatively simple proof of this result. Our proof uses Section 3, generalized to the bounded-genus case, and forbidding lakes.

We need the following theorem from the contraction-bidimensionality theory, and a simple corollary.

Theorem 11 [24] *There is a sequence of contractions that brings any graph G of genus g to a partially triangulated $\Omega(\mathrm{tw}(G)/(g+1)) \times \Omega(\mathrm{tw}(G)/(g+1))$ grid augmented with at most g additional edges.*

Corollary 12 *There is a sequence of contractions that brings any graph G of genus g to a partially triangulated $\Omega(\mathrm{tw}(G)/(g+1)^2) \times \Omega(\mathrm{tw}(G)/(g+1)^2)$ grid, augmented with at most g additional edges incident only to boundary vertices of the grid.*

Proof: We take the augmented $\Omega(\mathrm{tw}(G)/(g+1)) \times \Omega(\mathrm{tw}(G)/(g+1))$ grid guaranteed by Theorem 11, and find the largest square subgrid that does not contain in its interior any endpoints of the at most g additional edges. This subgrid has size $\Omega(\mathrm{tw}(G)/(g+1)^2) \times \Omega(\mathrm{tw}(G)/(g+1)^2)$ because there are $2g$ vertices to avoid. Then we contract all vertices outside this subgrid into the boundary vertices of this subgrid. □

The main idea for proving a relation between the treewidth of a graph and the treewidth of its dual is to relate both to the treewidth of the radial graph, and use that the radial graph of the primal is equal to the radial graph of the dual.

Theorem 13 *For a 2-connected graph G 2-cell embedded in a surface of genus g, its treewidth is within an $O((g+1)^2)$ factor of the treewidth of its radial graph $R(G)$.*

Proof: We follow the part of the proof of Lemma 4 establishing that $\mathrm{tw}(G)+1 = \Omega(\mathrm{tw}(R \cup G)+1)$, in order to prove that $\mathrm{tw}(G)+1 = \Omega(\mathrm{tw}(R)+1)$. The differences are as follows. Every occurrence of $R \cup G$ is replaced by R. Instead of applying Theorem 1 to obtain a grid minor K and then discarding the edge deletions from the sequence to obtain a partially triangulated grid contraction K', we use Corollary 12 to obtain a partially triangulated $\Omega(\mathrm{tw}(R)/(g+1)) \times \Omega(\mathrm{tw}(R)/(g+1))$ grid contraction K' of R augmented with at most g additional edges incident only to boundary vertices of the grid. Otherwise, the proof is identical, and we obtain an $\Omega(\mathrm{tw}(R)/(g+1)^2) \times \Omega(\mathrm{tw}(R)/(g+1)^2)$ grid contraction K'' of G. Therefore, $\mathrm{tw}(G)+1 = \Omega(\mathrm{tw}(R)/(g+1)^2)$. Because G is 2-connected, $\mathrm{tw}(G) > 0$, so $\mathrm{tw}(G) = \Omega(\mathrm{tw}(R)/(g+1)^2)$.

Now we apply what we just proved—$\mathrm{tw}(G) = \Omega(\mathrm{tw}(R(G))/(g+1)^2)$—substituting $R(G)$ for G. (The theorem applies: $R(G)$ is 2-cell embeddable in the same surface as G, and $R(G)$ is 2-connected because G (and thus G^*) is 2-connected.) Thus $\mathrm{tw}(R(G)) = \Omega(\mathrm{tw}(R(R(G)))/(g+1)^2)$. We claim that G is a minor of $R(R(G))$, which implies that $\mathrm{tw}(G) \leq \mathrm{tw}(R(R(G)))$ and therefore $\mathrm{tw}(R(G)) = \Omega(\mathrm{tw}(G)/(g+1)^2)$ as desired.

Now we prove the claim. Because G is 2-connected, each face of the radial graph $R(G)$ is a diamond (4-cycle) v_1, f_1, v_2, f_2 alternating between vertices (v_1 and v_2) and faces (f_1 and f_2) of G. Also, $v_1 \neq v_2$ and $f_1 \neq f_2$. If we take the radial graph of the radial graph, $R(R(G))$, we obtain a new vertex w for each such diamond, connected via edges to v_1, f_1, v_2, and f_2. For each such vertex w, we delete the edges $\{w, f_1\}$ and $\{w, f_2\}$, and we contract the edge $\{w, v_2\}$. The local result is just the edge $\{v_1, v_2\}$. Overall, we obtain G as a minor of $R(R(G))$. □

With this connection to the radial graph in hand, we can prove the main theorem of this section:

Theorem 14 *The treewidth of a graph G 2-cell embedded in a surface of genus g is at most $O(g^4)$ times the treewidth of the dual G^*.*

Proof: If G is 2-connected, then by Theorem 13, $\mathrm{tw}(G)$ is within an $O(g^2)$ factor of $\mathrm{tw}(R(G))$. Because $R(G^*) = R(G)$, we also have that $\mathrm{tw}(G^*)$ is within an $O(g^2)$ factor of $\mathrm{tw}(R(G))$. Therefore, $\mathrm{tw}(G)$ is within an $O(g^4)$ factor of $\mathrm{tw}(G^*)$.

Now suppose G has a vertex 1-cut $\{v\}$. Then G has two strictly smaller induced subgraphs G_1 and G_2 that overlap only at vertex v and whose union $G_1 \cup G_2$ is G. The treewidth of G is the maximum of the treewidth of G_1 and the treewidth of G_2. (Given tree decompositions of G_1 and G_2, pick a node in each tree whose bag contains v, and connect these nodes together via an edge.) Furthermore, the dual graph G^* has a cut vertex f corresponding to v, and G^* similarly decomposes into induced subgraphs G_1^* and G_2^* such that $G_1^* \cup G_2^* = G^*$ and G_1^* and G_2^* overlap only at f. By induction, $\mathrm{tw}(G_i)$ is within a cg^4 factor of $\mathrm{tw}(G_i^*)$, for $i \in \{1, 2\}$ and for a fixed constant c. Therefore, $\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$ is within a cg^4 factor of $\max\{\mathrm{tw}(G_1^*), \mathrm{tw}(G_2^*)\}) = \Theta(\mathrm{tw}(G^*))$. □

The bound in Theorem 14 is not necessarily the best possible. In particular, we can improve the bound from $O(g^4)$ to $O(g^2)$. Instead of using Corollary 12, we can apply Theorem 11 directly and instead modify the grid argument of Lemma 4 to avoid the endpoints of the g additional edges. Specifically, we stretch the "waffle" of horizontal and vertical strips in the grid connecting the $v_{i,j}$'s, so that all grid points we use for paths avoid all rows and columns containing the endpoints of the g additional edges. Then we can use the same argument, deleting the vertices and edges not on the paths, and in particular deleting the g additional edges, to form the desired grid minor.

7 Improved Grid Minor Bounds for $K_{3,k}$

Recall that every graph excluding a fixed minor H having treewidth at least $c_H r$ has the $r \times r$ grid as a minor [18]. The main result of this section is an explicit bound on c_H when $H = K_{3,k}$ for any k (proved in Section 7.1):

Theorem 15 *Suppose G is a graph with no $K_{3,k}$-minor. If the treewidth is at least $20^{4k} r$, then G has an $r \times r$ grid minor.*

In [55], it was shown that, if the treewidth is at least $f(r) \geq 20^{2^r}$, then G has an $r \times r$ grid as a minor. Our second theorems use this result to show the following. A *separation* of G is an ordered pair (A, B) of subgraphs of G such that $A \cup B = G$ and there are no edges between $A - B$ and $B - A$. Its *order* is $|A \cap B|$. Suppose G has a separation (A, B) of order k. Let A^+ be the graph obtained from A by adding edges joining every pair of vertices in $V(A) \cap V(B)$. Let B^+ be obtained from B similarly. We say that G is a *k-sum* of A^+ and B^+. If both A^+ and B^+ are minors of G other than G itself, we say that G is a *proper k-sum* of A^+ and B^+.

Using similar techniques as the theorem above, we prove the following two structural results decomposing $K_{3,4}$-minor-free and K_6^--minor-free graphs into proper k-sums (proved in Section 7.2):

Theorem 16 *Every $K_{3,4}$-minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most* $20^{2^{15}}$.

Theorem 17 *Every K_6^--minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most* $20^{2^{15}}$.

These theorems are explicit versions of the following decomposition result for general single-crossing-minor-free graphs (including $K_{3,4}$-minor-free and K_6^--minor-free graphs):

Theorem 18 [52] *For any fixed single-crossing graph H, there is a constant w_H such that every H-minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most w_H.*

This result heavily depends on Graph Minor Theory, so the treewidth bound w_H is huge—in fact, no explicit bound is known. Theorems 16 and 17 give reasonable bounds for the two instances of H we consider. We prove Theorems 16 and 17 simultaneously, because the proofs are almost the same. Our proof of Theorems 16 and 17 uses a 15×15 grid minor together with the result in [51]. The latter result says roughly that, if there is a planar subgraph H in a nonplanar graph G, then H has either a nonplanar "jump" or "cross" in G such that the resulting graph is a minor of G. Our approach is to find a $K_{3,4}$-minor and a K_6^--minor in a 15×15 grid minor plus some nonplanar jump or cross.

7.1 Proof of Improved Grid Minor Bounds for $K_{3,k}$

In this section, we prove Theorem 15. First we need some definitions and basic lemmas.

We call a set $X \subseteq V(G)$ *k-connected* in G if $|X| \geq k$ and, for all subset $Y, Z \subseteq X$ with $|Y| = |Z| \leq k$, there are $|Y|$ disjoint paths in G from Y to Z. Note that the sets Y and Z are not required to be disjoint. If $X = G$, then we say G is k-connected. The set X is *externally k-connected* if, in addition, the required paths do not contain any vertex in X except their endpoints. For example, the vertex set of any k-connected subgraph of G is k-connected in G (though not necessarily externally), and any horizontal path of the $k \times k$ grid is k-connected in the grid, even externally.

Following [27], call a separation (A, B) a *premesh* if all the edges of $A \cap B$ lie in A, and A contains a tree T with the following properties:

1. T has maximum degree at most 3;

2. every vertex of $A \cap B$ lies in T and has degree at most 2 in T; and

3. T has a leaf in $A \cap B$.

A premesh (A, B) is a *k-mesh* if $V(A \cap B)$ is externally k-connected in B, and the graph $G = A \cup B$ is said to *have* this premesh or k-mesh.

Among useful lemmas on the k-mesh, Diestel et al. [27] proved the following lemma.

Lemma 19 [27, Lemma 4] *Let G be a graph and let $h \geq k \geq 1$ be integers. If G has no k-mesh of order h, then G has treewidth less than $h + k - 1$.*

Therefore, if the treewidth of G is at least $h + k$, then G has a k-mesh of order h.

Lemma 20 (see [26, 27]) *Let $k \geq 2$ be an integer, let T be a tree of maximum degree at most 3, and let $X \subseteq V(T)$ with $|X| \geq k$. Then T has an edge set $E \subseteq E(G)$ such that every component of $T - E$ has at least k vertices and at most $2k - 2$ vertices in X, except that one such component may have fewer vertices in X.*

Lemma 21 [26] *Let G be a bipartite graph with bipartition (A, B) with $|A| = a$ and $|B| = b$. Let $c \leq a$ and $d \leq b$ be positive integers. Assume that G has at most $(a-c)(b-d)/d$ edges. Then there exist $C \subseteq A$ and $D \subseteq B$ such that $|C| = c$, $|D| = d$, and $C \cup D$ is independent in G.*

A *linkage* L is a set of disjoint paths in a graph. L is an *A–B linkage* if each member is a path from A to B. Let $|L|$ denote the number of paths.

The following is our key lemma. We follow some parts of the proof by Diestel et al. [26]. The proof of (2) below was simplified by R. Thomas (personal communication, 2005).

Lemma 22 *Suppose $\mathbf{P}$ and $\mathbf{Q}$ are an A–B linkage and a C–D linkage, respectively, where A, B, C, and D are pairwise disjoint, and $|\mathbf{P}| = 20r$, $|\mathbf{Q}| \geq 24320kr$. Assume further that all the paths in $\mathbf{Q}$ meet all but at most r paths in $\mathbf{P}$. Let G be the graph consisting of the linkages $\mathbf{P}$ and $\mathbf{Q}$. Suppose that, for any $e \in \bigcup_{P \in \mathbf{P}} E(P) - \bigcup_{Q \in \mathbf{Q}} E(Q)$, the graph $G - e$ does not have an A–B linkage of size $20r$. We call this property "minimality". Then either*

1. *G has a $K_{3,k}$-minor, or*
2. *G has an $r \times r$ grid minor.*

Proof: Let us first introduce notation for analyzing sets of paths in a graph. Let $P_1, \ldots, P_q$ be vertex disjoint paths, and let $Z = P_1 \cup \ldots \cup P_q$ be a linkage. Recall that a *Z-bridge* in G is either an edge $e \in E(G) \backslash E(Z)$ whose endpoints are both in Z, or a subgraph of G consisting of a connected component C of $G - Z$ together with all edges joining C and Z. The vertices of a Z-bridge B in $Z \cap B$ are called *attachements* of B, and we say that B is *attached* to Z at these vertices. Given any two subpaths P and Q contained in the linkage Z, we say that they are *adjacent* if there exists a Z-bridge whose attachments are in both P and Q.

Our first goal is to prove the following:

(1) There exist $X \subseteq \mathbf{P}$ and $Y \subseteq \mathbf{Q}$ such that $|X| = 4$ and $|Y| \geq 3|\mathbf{Q}|/4$, and each path in Y intersects all the paths in X.

Let W be the bipartite graph with bipartition $(\mathbf{P}, \mathbf{Q})$ in which $P \in \mathbf{P}$ is adjacent to $Q \in \mathbf{Q}$ whenever $P \cap Q = \emptyset$. We claim that, in W, there exist $X \subseteq \mathbf{P}$ and $Y \subseteq \mathbf{Q}$ such that $|X| = 4$, $|Y| \geq 3|\mathbf{Q}|/4$, and $X \cup Y$ are independent in W. The bipartite graph W has at most $|\mathbf{Q}|r$ edges. By Lemma 21 with $a = |\mathbf{Q}|, c = 3|\mathbf{Q}|/4, b = 20r$, and $d = 4$, we have the desired X and Y. This implies that each path in Y intersects all the paths in X. This proves (1).

Hereafter, we assume $X = \{P_1, P_2, P_3, P_4\}$, and other paths of $\mathbf{P}$ are $P_5, \ldots, P_{20r}$.

Our next goal is to prove the following.

(2) Each path $P_j \in \mathbf{P}$ has vertices $p_{j,i}$ for $i = 1, \ldots, 228k$ (not necessarily disjoint), which appear in this order, such that the following holds. Define the *ith segment* of P_j to be the subpath of P_j between $p_{j,i}$ and $p_{j,i+1}$. Then there is a subset $Q' \subset Y$ with $|Q'| \geq 40r \times 228k$ such that each path in Q' meets all but at most r paths of $\mathbf{P}$ only in their ith segments, for some i with $1 \leq i \leq 228k$. Moreover, there are at least $40r$ paths in Q' that stay strictly inside the union of ith segments of $\mathbf{P}$.

Walk along P_1 from A until having encountered $80r$ paths in Y, then pick up $e_1 \in E(P_1) - \bigcup_{Q \in \mathbf{Q}} E(Q)$. Then walk along P_1 until having encountered another $80r$ paths in Y, then pick up $e_2 \in E(P_1) - \bigcup_{Q \in \mathbf{Q}} E(Q)$, and so on. Hence we pick up such edges at least $228k$ times because $|Y| \geq 3|\mathbf{Q}|/4 \geq 18240kr$.

By the assumption of "minimality" and Menger's theorem, there exists a cutset of size at most $20r-1$ separating A and B in $G-e_i$ for each i. Clearly each path P_j contains exactly one vertex in this cutset for $2 \le j \le 20r$. Let $\{p_{2,i},\dots,p_{20r,i}\}$ be the set of vertices consisting of the cutset in $G-e_i$ such that P_j contains $p_{j,i}$ for $2 \le j \le 20r$. We may define $p_{1,i}$ as one of vertices of e_i. Let us define the segment $P_j[i,i+1]$ to be the subpath of P_j between p_i and p_{i+1}, for $i=1,\dots,228k-1$. Note that some $P_j[i,i+1]$'s could be a single vertex. The vertex set $\{p_{1,i},\dots,p_{20r,i}\}$ divides $\mathbf{P}$ into two parts P^{R_i} and P^{L_i} such that P^{R_i} consists of the linkages from A to $\{p_{1,i},\dots,p_{20r,i}\}$, and P^{L_i} consists of the linkages from B to $\{p_{1,i},\dots,p_{20r,i}\}$, respectively. Let us remind that at least $80r$ paths in Y hit $P_1[i,i+1]$.

Define the *ith interval* to be $\bigcup_{j=1}^{20r} P_j[i,i+1]$. We claim that at least $80r-40r$ of the $80r$ paths in Y encountered on $P_1[i,i+1]$ do not leave the ith interval. Hence at least $40r$ paths in Y stay strictly inside the ith interval. We first look at the $80r$ paths in Y encountered on $P_1[i,i+1]$. Because there is no path from A to B in $G-\{p_{i,1},\dots,p_{i,20r}\}$, at most $20r$ paths of the $80r$ paths in Y leave for $P^{R_i}-\{p_{1,i},\dots,p_{20r,i}\}$ through $\{p_{1,i},\dots,p_{20r,i}\}$. Similarly, at most $20r$ paths of the $80r$ paths in Y leave for $P^{L_{i+1}}-\{p_{1,i+1},\dots,p_{20r,i+1}\}$ through $\{p_{1,i+1},\dots,p_{20r,i+1}\}$. Therefore, at least $80r-40r$ of the $80r$ paths in Y encountered on $P_1[i,i+1]$ do not leave the ith interval. This proves (2).

Thus these cutsets $\{p_{1,i},\dots,p_{20r,i}\}$ for $1 \le i \le 228k$ will break the elements of $\mathbf{P}$ into intervals; see Figure 2. Moreover, each interval contains at least $40r$ paths in Y that stay strictly inside the interval. Note that some paths of $\mathbf{P}$ in the ith interval might be a singleton, but there are at most r such vertices.

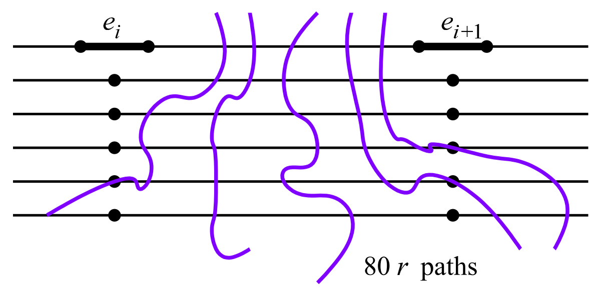

Figure 2: Paths in the proof of Lemma 22.

Let us introduce some further definitions. For each ith interval, define H_i to be the auxiliary graph with vertex set $X = P_1,P_2,P_3,P_4$ such that P_j and $P_{j'}$ are adjacent in H_i if there is a bridge with attachments both in P_j and $P_{j'}$ with $j \neq j'$ in the ith interval. Define H'_i to be the graph in the ith interval. Here by "bridges" we refer to the bridges for the linkage P_1,P_2,P_3,P_4 consisting of only the paths in $\mathbf{P}-X$ and the paths in Y each of which stays strictly inside H'_i. We also assume that H'_i consists of the paths in $\mathbf{P}$ and the paths in Y each of which stays strictly inside the ith interval. Note that $|H'_i \cap H'_{i+1}| = 20r$ for each i, and because at least $40r$ of the $80r$ paths in Y stay strictly inside the ith interval and they hit all of P_1,P_2,P_3,P_4, so all of the H_i's are connected (and have four vertices) and none of the P_j's are trivial paths in H'_i for $j=1,2,3,4$.

Our idea for using these auxiliary graphs is the following.

(3) If there are $4k$ H_i's, each of which has a vertex of degree 3 in H_i, then we can find a $K_{3,k}$-minor.

This is because at least k H'_i's have bridges with attachments both in P_j and $P_{j'}$ for some j and for any $j' \neq j$. Therefore, contracting each $P_{j'}$ with $j' \neq j$ into a single vertex together with P_j would give rise to a $K_{3,k}$-minor. This argument is due to [5, 3]. (Actually, the rest of the proof is inspired by the argument in [5, 3].)

Hence by (3), at least $224k$ H_i's are either a 4-vertex path or a 4-vertex cycle. (One would normally write these graph types as P_4 and C_4, but this notation would conflict with the existing notion of P_4.) We now look at every seven consecutive H'_i's, e.g., $\bigcup_{j=i}^{i+6} H'_j$ for $i=1,8,\dots,224k-6$, and its auxiliary graph H''_i. There are at least $32k$ H''_i's. (We shall define H''_i later, but one can imagine that it is the auxiliary graph of $\bigcup_{j=i}^{i+6} H'_j$, which is similar to the definition of the auxiliary graph H_i for H'_i. So H''_i consists of the four vertices.)

In the rest of the proof, we shall prove that either one of the $224k$ H'_i is planar or we can make the auxiliary graph H''_i of $\bigcup_{j=i}^{i+6} H'_j$ (for $i = 1, 8, \ldots, 224k-6$) to have a vertex of degree 3. If the first happens, then we will find an $r \times r$-grid minor. If the second happens, then we can find a $K_{3,k}$-minor by (3) and because there are at least $32k$ H''_i's.

Our next goal is to prove that, if the first happens, then there is an $r \times r$-grid minor.

(4) For each i, H'_i does not result in a planar graph, up to 3-separations.

We shall prove that, otherwise, it would contain an $r \times r$ grid minor, a contradiction.

Suppose H_i is planar, up to 3-separations, and we can label $\mathbf{P} = P_1, \ldots, P_{20r}$ in such a way that $P_1, \ldots, P_{20r}$ appear in the clockwise order in H_i.

Suppose $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$ are 3-separations. So $|B_i - A_i|, |A_i - B_i| \geq 2$. Let R be at least $40r$ paths of Y that stay strictly inside H'_i. Because each path of R hits all of P_1, P_2, P_3, P_4, we can assume that B_i always contains P_1, P_2, P_3, P_4, and hence it hits a large part of each path in R. We claim that, after putting the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$, removing all the graphs in $A_i - B_i$ and furthermore, letting F be the resulting planar graph, there are at least $19r$ paths of $\mathbf{P}$ in F and all paths of R are in F. Otherwise, because each B_i contains a large part of each path in R, some of the paths in R cannot hit $19r$ paths of $\mathbf{P}$. Note that some of the paths in R may go through the edges of the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$.

We now put the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$, and remove all the graphs in $A_i - B_i$. Then the resulting graph F is planar.

We may assume that F contains $P_1, \ldots, P_{19r}$ of $\mathbf{P}$. By possibly relabeling $\mathbf{P}$, we may assume that $P_1, \ldots, P_{19r}$ of $\mathbf{P}$ appear in this order in F. Moreover, we add edges $p_{j,i}p_{j,i+1}$ and $p_{j,i+1}p_{j+1,i+1}$ for $j = 1, \ldots, 19r-1$. We first consider the case that there is no bridge whose attachments are in both P_1 and P_{19r}. Therefore, now F has an outer cycle C that consists of P_1, P_{19r}, and edges $p_{j,i}p_{j,i+1}$ and $p_{j,i+1}p_{j+1,i+1}$ for $j = 1, \ldots, 19r-1$.

For each path P in R, we only consider a subpath P' of P that starts at a vertex in P_i and ends at a vertex in either in P_{i+18r} or P_{i-18r}. Because each path in R hits all but at most r paths of $P_1, \ldots, P_{19r}$ in $\mathbf{P}$ and F is planar, we can take such a subpath of each path in R. Let R' be the set of the subpath for each path in R.

We now claim that there are at least $40r$ paths in R' such that each of them appears in the order $P_j, P_{j+1}, \ldots, P_{j+18r-1}$ or in the order $P_j, P_{j-1}, \ldots, P_{j-18r+1}$ (mod $19r$) for some j with $1 \leq j \leq 19r$ (by possibly rerouting some paths of R' through the linkage $P_1, \ldots, P_{19r}$ of $\mathbf{P}$) and does not cross over other paths in F. Because at least $40r$ paths of R' stay strictly inside F, and F is now planar, they occur in such a way that each path P in R' sees any other path in R' in either the left side of P or the right side of P. Although some of the paths in R' may hit some paths of $\mathbf{P}$ more than twice, it is possible that, by possibly rerouting some of the paths in R' through the linkage $P_1, \ldots, P_{19r}$ of $\mathbf{P}$, each of at least $40r$ paths in R' appears in the order $P_j, P_{j+1}, \ldots, P_{j+18r-1}$ or in the order $P_j, P_{j-1}, \ldots, P_{j-18r+1}$ (modulo $19r$) for some j with $1 \leq j \leq 20r$. In fact, we can reroute the paths so that the intersection of P_i and each path in R' is a path. We refer the reader to the proof of this claim in [55, 45, 47] (the argument showing that any planar graph G with treewidth at least $6r$ has an $r \times r$ grid minor; also, (10.1) in [47]). But let us sketch the argument. We take R' and $\mathbf{P}$ such that $|E(R') - E(\mathbf{P})|$ is as small as possible. Because there is no subpath P of a path in R' such that P starts at a vertex u in P_j and ends at some other vertex v in P_j with $P \cap \bigcup_{x=1}^{19r} P_x = \{u, v\}$ by the minimality, if we take a lowest peak in R', then we can always reroute a path through the linkage of $P_1, \ldots, P_{19r}$ of $\mathbf{P}$. This would contradict the minimality of $|E(R') - E(\mathbf{P})|$.

We claim that these $40r$ paths in R' together with $P_1, \ldots, P_{19r}$ of $\mathbf{P}$ would give rise to an $r \times r$ grid minor. Note that each path in R' intersects all but at most r paths in $\mathbf{P}$. Let x_j be the number of paths in R' each of which hits P_j first. So these x_j paths intersect $P_1, \ldots, P_{19r}$ of $\mathbf{P}$ in the order $P_j, P_{j+1}, \ldots, P_{j+18r-1}$ or in the order $P_j, P_{j-1}, \ldots, P_{j-18r+1}$ (modulo $19r$). Then for some j with $1 \leq j \leq 19r$, if $\sum_{y=j}^{j+17r} x_y \geq 2r$, then clearly there is an $r \times r$ grid minor. Hence this does not happen. So $\sum_{j=1}^{20r} \sum_{y=j}^{j+17r} x_y < 40r^2$. On the other hand, the above inequality counts each path of R' exactly $17r$ times. Therefore, $17r \cdot 40r < 40r^2$, a contradiction. Hence, for each i, H'_i is not planar. Actually, the argument still works if at least $2r$ paths of $\mathbf{Q}$ hit all but at most r of the cr paths of $\mathbf{P}$, for $c \geq 3$. We shall use this fact later.

Finally suppose that there is a bridge whose attachements are in P_1 and P_{19r}. In this case, edges $p_{j,i}p_{j,i+1}$ for $j = 1, \ldots, 19r$ consists of the outer face boundary of F, and edges $p_{j,i+1}p_{j+1,i+1}$ for $j = 1, \ldots, 19r$ consists of a face in F. Note that we also add the edges $p_{19r,i}p_{1,i}, p_{19r,i+1}p_{1,i+1}$. Again we consider the set R' that consists of a subpath of each path in R, as described above. So each path in R intersects $P_1, \ldots, P_{19r}$ of $\mathbf{P}$ in the order $P_j, P_{j+1}, \ldots, P_{j+18r-1}$ or in the order $P_j, P_{j-1}, \ldots, P_{j-18r+1}$ (modulo $19r$). Let x'_j be the number of paths in R' each of which does not hit P_j. We now delete the path P_y such that x'_y is maximum. We claim $x'_y \geq 2r$. Suppose not. Then $\sum_{j=1}^{19r} x'_j \leq 2r \cdot 19r$. On the other hand, this inequality counts each path in R' exactly r times. So, $2r \cdot 19r \geq 40r \cdot r$, a contradiction. Hence $x'_y \geq 2r$. After deleting the path P_y, we can relabel the $19r - 1$ paths of $\mathbf{P}$ in F so that they appear in the order $P_1, \ldots, P_{19r-1}$. Also there are at least $2r$ paths in R' each of which hits all but at most r paths in $P_1, \ldots, P_{19r-1}$. Therefore, by the above remark, we can find an $r \times r$ grid minor. This proves (4).

Our next goal is to analyze the bridge structure in $\bigcup_{j=i}^{i+6} H'_j$ (for $i = 1, 8, \ldots, 224k - 6$). By (4), we may assume that each H'_i is not planar, up to 3-separations.

Because there are at least $224k$ H_i's that are either 4-vertex paths or 4-vertex cycles, by the Pigeonhole Principle, at least $56k$ of these $224k$ H_i's have the same 4-vertex path with vertex set $\{P_1, P_2, P_3, P_4\}$. More precisely, we may assume that at least $56k$ H_i's have the 4-vertex path consisting of the edges P_1P_2, P_2P_3, P_3P_4. Hereafter, we shall consider only these $56k$ H_i's each of which has the same 4-vertex path. For convenience of notation, we assume that each H_i with $i = 1, \ldots, 56k$ is either a 4-vertex path or a 4-vertex cycle, containing the edges between P_1 and P_2, P_2 and P_3, and P_3 and P_4.

We now define H''_i as the auxiliary graph in $H'_{7i-6} \cup H'_{7i-5} \cup H'_{7i-4} \cup H'_{7i-3} \cup H'_{7i-2} \cup H'_{7i-1} \cup H'_{7i}$. So the vertex set of H''_i is $\{P_1, P_2, P_3, P_4\}$. We know that, for each i, H''_i is either a 4-vertex path or a 4-vertex cycle, and there are $8k$ H''_i's.

As pointed out before (4), our goal in the rest of the proof is to show that we can make the graph H''_i such that H''_i has a vertex of degree 3. Because we have $8k$ H''_i's, by (3), we would get a $K_{3,k}$-minor, a contradiction. To prove that, we consider the following two cases.

Case 1. There are at least $4k$ H''_i's each of which is a 4-vertex path.

Case 2. There are at most $4k$ H''_i's each of which is a 4-vertex cycle.

Actually, Case 2 is much easier and simpler to deal with, and the proof just follows Case 1. So we just focus on Case 1, and leave the proof of Case 2 to the reader.

Let us look at one H''_i that is a 4-vertex path. So H_{7i-6}, H_{7i-5}, H_{7i-4}, H_{7i-3}, H_{7i-2}, H_{7i-1}, and H_{7i} are all 4-vertex paths. We may assume that the edges P_1P_2, P_2P_3, P_3P_4 are in all of $E(H_{7i-6})$, $E(H_{7i-5})$, $E(H_{7i-4})$, $E(H_{7i-3})$, $E(H_{7i-2})$, $E(H_{7i-1})$, and $E(H_{7i})$.

We now look at H'_{7i-3}. Let $Z_{j,7i-3}$ be the graph consisting of all the bridges attached to both P_j and P_{j+1} in H'_{7i-3} for $j = 1, 2, 3$. Because H_{7i-3} is a 4-vertex path, the bridges in $Z_{j,7i-3}$ have attachments only in P_j and P_{j+1} for $j = 1, 2, 3$. Let $Z'_{j,7i-3}$ be the union of bridges that attach only to P_j in H'_{7i-3} for $j = 1, 2, 3, 4$. Let $Z'_{7i-3} = \bigcup_{j=2}^{3} Z'_{j,7i-3}$.

The reason why we focus on H'_{7i-3} is that we can easily "extend" H'_{7i-3} by involving H'_{7i-2}, H'_{7i-4}, or even H'_{7i-1}, H'_{7i-5}, to find bridges that attach to some two of P_1, P_2, P_3, P_4.

Informally, the rest of argument in our proof proceeds as follows. If we can make a cross in $Z_{2,7i-3} \cup P_2 \cup P_3$, that is, if there are two disjoint paths P'_2 and P'_3 such that P'_2 starts at $p_{2,7i-3}$ and ends at $p_{3,7i-2}$, and P'_3 starts at $p_{3,7i-3}$ and ends at $p_{2,7i-2}$, then we are happy. This is because we can make the graph H''_i so that some vertex has degree 3. To see this, suppose we can make a cross in $Z_{2,7i-3} \cup P_2 \cup P_3$. Then P_3 has a bridge attached to both P_3 and P_1 in H'_{7i-2} and P_3 has a bridge attached to both P_3 and P_j for $j = 2, 4$ in H'_{7i-4}. Hence P_3 has degree 3 in H''_i. But if we cannot make a cross in $Z_{2,7i-3} \cup P_1 \cup P_2$, then $Z_{2,7i-3} \cup P_2 \cup P_3$ is planar (with the outer face boundary containing P_2, P_3), up to 3-separations, by Seymour's theorem or [47] (or see [5] for details.).

Similarly, we can make a cross in $Z'_{1,7i-3} \cup Z_{1,7i-3} \cup P_1 \cup P_2$ unless $Z'_{1,7i-3} \cup Z_{1,7i-3} \cup P_1 \cup P_2$ is planar (with the outer face boundary containing P_2), up to 3-separations. Moreover, we can make a cross in $Z'_{4,7i-3} \cup Z_{3,7i-3} \cup P_3 \cup P_4$ unless $Z'_{4,7i-3} \cup Z_{3,7i-3} \cup P_3 \cup P_4$ is planar (with the outer face boundary containing P_3), up to 3-separations. Hence, if Z'_{7i-3} were empty, it would be a contradiction by (4) because H'_{7i-3} would be planar, up to 3-separations. On the other hand, we could not prove that Z'_{7i-3} is empty. Instead, we shall prove that the number of paths in $\mathbf{P}$ that hit Z'_{7i-3} is at most r. Finally, after deleting the graph Z'_{7i-3}, the resulting graph F is planar, up to 3-separations. Because F contains at least $19r$ paths of $\mathbf{P}$, and there are at least $40r$ paths that hit all but at most r paths of these $19r$ paths, by a similar argument to the proof of (4), we can find an $r \times r$ grid minor, a contradiction.

Therefore, our next goal is to prove the property of Z'_{7i-3}. Let us remind the reader that, in the rest of the argument, we are trying to make the graph H''_i so that it has a vertex of degree 3.

We claim the following:

(5) Z'_{7i-3} intersects at most r paths in $\mathbf{P}$.

Let P'' be the set of paths in $\mathbf{P}$ each of which intersects Z'_{7i-3}. Actually, each path of P'' in H'_{7i-3} is strictly contained in Z'_{7i-3}. Suppose for contradiction that $|P''| \geq r + 1$. Because all the paths in Y hit all but at most r paths in $\mathbf{P}$, we may assume that there is a path $P' \in P''$ that intersects $Z'_{2,7i-3}$ in such a way that, in H_{7i-5}, there is a path W_1 from P' to $P_1 \cup P_2 \cup P_3 \cup P_4$, and in addition, in H_{7i-1}, there is a path W_2 from P' to $P_1 \cup P_2 \cup P_3 \cup P_4$. If either W_1 or W_2 arrives on P_4, then clearly P_2 has degree 3 in H''_i.

Next, we consider the cases that both W_1 and W_2 arrive on either P_2 or P_1. If W_1 arrives on P_1 and W_2 arrives on P_2, then we can cross P_1 and P_2 because there is a path between P_1 and P_2 in $Z_{1,7i-3}$. Similarly, we can cross P_1 and P_2 when W_1 arrives on P_2 and W_2 arrives on P_1. In both cases, it is easy to see that P_3 becomes degree 3 in H''_i (together with bridges in H'_{7i} and in H'_{7i-6}). Suppose both W_1 and W_2 arrive on P_2. Then we can replace the path P_2 in $H'_{7i-1} \cup H'_{7i-2} \cup H'_{7i-3} \cup H'_{7i-4} \cup H'_{7i-5}$ by W_1, W_2 and the part of P'. Then clearly P_3 has degree 3 in H''_i because P_2 in H'_{7i-3} now becomes a bridge with attachments to P_1 and the new P_2 and P_3.

Suppose that both W_1 and W_2 arrive on P_1. Replace the path P_1 in $H'_{7i-1} \cup H'_{7i-2} \cup H'_{7i-3} \cup H'_{7i-4} \cup H'_{7i-5}$ by W_1, W_2 and the part of P'. We take a path W_3 in $Z_{1,7i-2}$ that joins P_1 and P_2. If W_3 intersects P', then we can reduce to the case that W_1 arrives on P_1 and W_2 arrives on P_2 in the previous paragraph because it does not require that W_1 be in H'_{7i-5} nor that W_2 be in H'_{7i-1}. Also, we take a path W_4 in $Z_{1,7i-4}$ that joins P_1 and P_2. By the above observation, we may assume that W_4 does not intersect P'. Replace P_2 in $H'_{7i-4} \cup H'_{7i-3} \cup H'_{7i-2}$ by W_3, W_4 and the part of P_1 in $H'_{7i-4} \cup H'_{7i-3} \cup H'_{7i-2}$. Then clearly P_3 has degree 3 in H''_i because P_2 in H'_{7i-3} now becomes a bridge with attachments to the new P_1 and the new P_2 and P_3.

Finally suppose that either W_1 or W_2 arrives on P_3, say, W_1 arrives on P_3. We take a path W' in $Z_{2,7i-4}$ joining P_2 and P_3. If W' does not hit P', then we can reroute P_2 by following the original P_2 from $p_{2,7i-6}$, then following W' in H_{7i-4}, and finally following P_3 to $p_{2,7i}$. We can also reroute P_3 by following the original P_3 from $p_{3,7i-6}$, then following W_1 and P', and then following a path between P' and P_2 in $Z'_{2,7i-3}$, and finally following P_2 to $p_{2,7i}$. Because there is a bridge attached to both P_3 and P_4 in H'_{7i-6}, there is a bridge attached to both P_3 and P_2 in H'_{7i-6}, and furthermore, there is a bridge attached to both P_1 and P_3 in H'_{7i-2}, these reroutes of P_2, P_3 result in making the graph H''_i so that P_3 has degree 3 in H''_i. Suppose W' hit W_1. We now consider W_2. If W_2 arrives on either P_1 or P_2, then we can reduce to the case that W_1 arrives on P_2 and W_2 arrives on P_2 or W_1 arrives on P_2 and W_2 arrives on P_1 in the previous paragraph. So, finally suppose W_2 arrives on P_3. As we did for W', we take a path W'' in $Z_{2,7i-2}$ joining P_2 and P_3. If W'' does not hit P', then we can reduce to the case that we considered for W_1. If it hits, we can reduce to the case that W_1 arrives on P_2 and W_2 arrives on P_2 in the previous paragraph.

This proves (5).

We now prove that each $Z_{j,7i-3} \cup P_j \cup P_{j+1}$ in H'_{7i-3} is planar, up to 3-separations.

(6) For $j = 1, 2, 3$, $Z_{j,7i-3} \cup P_j \cup P_{j+1}$ in H'_{7i-3} is planar, up to 3-separations. Moreover, P_j, P_{j+1} in H'_{7i-3} are in the outer face boundary of this planar embedding.

For our convenience, we add edges $p_{j,7i-3}p_{j+1,7i-3}$ and $p_{j,7i-2}p_{j+1,7i-2}$ for $j = 1, 2, 3$. Suppose $Z_{1,7i-3} \cup P_1 \cup P_2$ in H'_{7i-3} is not planar, up to 3-separations. Note that $P_1 \cup P_2$ together with the edges $p_{1,7i-3}p_{2,7i-3}$ and $p_{1,7i-2}p_{2,7i-2}$ gives rise to a cycle C such that the graph $Z_{1,7i-3}$ is dividing, i.e., there is no path joining $Z_{1,7i-3}$ and a vertex outside C. Then by Seymour's theorem or [47] (or see [5] for more details), we can cross two paths P_1, P_2 in H'_{7i-3}. More precisely, there are two disjoint paths P'_1, P'_2 in H'_{7i-3} such that P'_1 joins $p_{1,7i-3}$ and $p_{2,7i-2}$, and P'_2 joins $p_{2,7i-3}$ and $p_{1,7i-2}$ in such a way that both P'_1 and P'_2 do not intersect any paths in P_3, P_4. We shall call these two paths a "cross".

Then P_3 has degree 3 in H''_i (together with bridges in H'_{7i-2} and in H'_{7i-4}). If we cannot make a cross, then, because all the bridges in $Z_{1,7i-3}$ are only attached to both P_1 and P_2 and the cycle C is dividing, by Seymour's theorem or [47] (or see [5] for more details), all the graphs in $Z_{2,7i-3} \cup P_1 \cup P_2$ can be embedded into the plane, up to 3-separations, such that C is the outer face boundary, and hence both P_1 and P_2 are on the outer face boundary.

Similarly, we can prove the cases for $j = 2, 3$. This proves (6).

We now prove one more property for Z'_{7i-3}.

(7) $Z'_{1,7i-3} \cup Z'_{2,7i-3} \cup Z_{1,7i-3} \cup P_1 \cup P_2$ and $Z'_{3,7i-3} \cup Z'_{4,7i-3} \cup Z_{3,7i-3} \cup P_3 \cup P_4$ in H'_{7i-3} are planar, up to 3-separations.

For our convenience, we add edges $p_{j,7i-3}p_{j+1,7i-3}$ and $p_{j,7i-2}p_{j+1,7i-2}$ for $j = 1, 2, 3$. Note that $P_1 \cup P_2$ together with the edges $p_{1,7i-3}p_{2,7i-3}$ and $p_{1,7i-2}p_{2,7i-2}$ gives rise to a cycle C such that the graph $Z_{1,7i-3}$ is dividing.

Suppose $Z'_{1,7i-3} \cup Z'_{2,7i-3} \cup Z_{1,7i-3} \cup P_1 \cup P_2$ in H'_{7i-3} is not planar, up to 3-separations. By (6), $Z_{1,7i-3} \cup P_1 \cup P_2$ is now planar, up to 3-separations, such that C is the outer face bounary.

Again, by Seymour's theorem or [47] (or see [5] for more details), we can cross two paths P_1, P_2 in H'_{7i-3}. More precisely, there are two disjoint paths P'_1, P'_2 in H'_{7i-3} such that P'_1 joins $p_{1,7i-3}$ and $p_{2,7i-2}$, and P'_2 joins $p_{2,7i-3}$ and $p_{1,7i-2}$ in such a way that both P'_1 and P'_2 do not intersect any paths in P_3, P_4.

Then P_3 has degree 3 in H''_i (together with bridges in H'_{7i-2} and in H'_{7i-4}). If we cannot make a cross, then, because all the bridges in $Z'_{1,7i-3}$ are only attached to P_1, all the bridges in $Z'_{2,7i-3}$ are only attached to P_2 and the graph $Z_{1,7i-3} \cup P_1 \cup P_2$ is planar, up to 3-separations, such that C is the outer face boundary of this embedding, and hence it is dividing, by Seymour's theorem or [47] (or see [5] for more details), all the graphs in $Z'_{1,7i-3} \cup P_1$ can be embedded into the plane, up to 3-separations, such that P_1 is on the outer face boundary, and all the graphs in $Z'_{2,7i-3} \cup P_2$ can be embedded into the plane, up to 3-separations, such that P_2 is on the outer face boundary. Therefore, it is easy to see that $Z'_{1,7i-3} \cup Z'_{2,7i-3} \cup Z_{1,7i-3} \cup P_1 \cup P_2$ in H'_{7i-3} is planar, up to 3-separations.

Similarly, we can prove the case for $Z'_{3,7i-3} \cup Z'_{4,7i-3} \cup Z_{3,7i-3} \cup P_3 \cup P_4$. This proves (7).

Let us observe that the proof of (7) implies that the graph $Z'_{1,7i-3} \cup P_1$ in H'_{7i-3} can be embedded into the plane, up to 3-separations, such that P_1 is in the outer face boundary. Similarly, the graph $Z'_{4,7i-3} \cup P_4$ in H'_{7i-3} can be embedded into the plane, up to 3-separations, such that P_4 is in the outer face boundary. Therefore, by (6) and (7), we get the following:

(8) The graph $\bigcup_{j=1}^{3} Z_{j,7i-3} \cup Z'_{1,7-3} \cup Z'_{4,7i-3} \cup \bigcup_{j=1}^{4} P_i$ is planar, up to 3-separations.

(8) does not imply that H'_{7i-3} is planar, up to 3-separations, because $Z'_{2,7i-3} \cup Z'_{3,7i-3}$ may not be empty. But they are both planar, up to 3-separations, by (7). In fact, the following claim shows that they do not create a cross.

(9) The graph $Z_{1,7i-3} \cup Z_{2,7i-3} \cup Z'_{2,7i-3} \cup P_1 \cup P_2 \cup P_3$ in H'_{7i-3} does not create a cross. More precisely, there are no two disjoint paths P', P'' such that P' joins $p_{2,7i-3}$ and $p_{2,7i-2}$ and P'' joins P_1 and P_3. Moreover, P' does not hit any paths of P_1, P_3, P_4 and P'' does not hit P_4 either.

Similarly, the graph $Z_{2,7i-3} \cup Z_{3,7i-3} \cup Z'_{3,7i-3} \cup P_2 \cup P_3 \cup P_4$ in H'_{7i-3} does not create a cross. More precisely, there are no two disjoint paths P', P'' such that P' joins $p_{3,7i-3}$ and $p_{3,7i-2}$ and P'' joins P_2 and P_4. Moreover, P' does not hit any paths of P_1, P_2, P_4 and P'' does not hit P_1 either.

Suppose that the first happens. Then P_3 is now adjacent to P_1. Together with a bridge attached to both P_2 and P_3 in H'_{7i-2} and a bridge attached to both P_3 and P_4 in H'_{7i-4}, P_3 now becomes degree 3 in H''_i. Similarly, we can prove the second case. This proves (9).

In the rest of the argument, we may not be able to make the graph H''_i so that it has a vertex of degree 3. But, instead, we shall find an $r \times r$ grid minor in H'_{7i-3}.

Let us look at the graph $Z = Z_{1,7i-3} \cup Z_{2,7i-3} \cup Z_{3,7i-3} \cup Z'_{1,7i-3} \cup Z'_{4,7i-3} \cup P_1 \cup P_2 \cup P_3 \cup P_4$. By (8), it is planar, up to 3-separations. By (5), it contains at least $19r$ paths of $\mathbf{P}$.

Suppose $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$ are 3-separations. So $|B_i - A_i|, |A_i - B_i| \geq 2$. Let R be at least $40r$ paths in Y that stay strictly in H'_i. Because each of R hits all of P_1, P_2, P_3, P_4, we can assume that B_i is always containing P_1, P_2, P_3, P_4, and hence it contains large part of each path in R. As in the proof of (4), after putting the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$, removing all the graphs in $A_i - B_i$ and furthermore, letting F be the resulting planar graph, there are at least $18r$ paths of $\mathbf{P}$ in F and all paths of R are in F. Note that some of the paths in R may go through the edges of the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$.

We now put the clique in $A_i \cap B_i$ for $i = 1, \ldots, l$, and remove all the graphs in $A_i - B_i$. Then the resulting graph F is planar.

We may assume that F contains $P_1, \ldots, P_{18r}$ of $\mathbf{P}$. By possibly relabeling $\mathbf{P}$, we may assume that $P_1, \ldots, P_{18r}$ of $\mathbf{P}$ appear in this order in F. Moreover, we add edges $p_{j,i}p_{j,i+1}$ and $p_{j,i+1}p_{j+1,i+1}$ for $j = 1, \ldots, 18r - 1$. We first consider the case that there is no bridge whose attachements are in both P_1 and P_{18r}. Therefore, now F has an outer cycle C that consists of P_1, P_{18r}, and edges $p_{j,i}p_{j,i+1}$ and $p_{j,i+1}p_{j+1,i+1}$ for $j = 1, \ldots, 18r - 1$.

For each path P in R, we only consider the subpath P' of each path in R such that P' starts at a vertex in P_i and ends at a vertex in either in P_{i+18r} or P_{i-18r}. Because each path in R hits all but at most r paths of $P_1, \dots, P_{18r}$ in $\mathbf{P}$ and F is planar, we can take such a subpath of each path in R. Let R' be the set of such a subpath of each path in R.

We now claim that there are at least $40r$ paths in R' such that, by possibly rerouting the paths in R' through the linkage $P_1, \dots, P_{18r}$ of $\mathbf{P}$, each of them appears in the order $P_j, P_{j+1}, \dots, P_{j+17r-1}$ or in the order $P_j, P_{j-1}, \dots, P_{j-17r+1}$ (mod $18r$) for some i with $1 \leq j \leq 18r$, and does not cross over other paths in H'_i. Note that some of the paths in R' may get in Z'_{7i-3}, but by (9), it is possible to reroute these paths through the linkage $P_1, \dots, P_{18r}$ of $\mathbf{P}$. Because at least $40r$ paths in R' stay strictly inside F, and F is now planar and (9) implies that there is no "jump", i.e, there is no cross over in H'_i, by the argument in the proof of (4), there are the desired $40r$ paths in R'. In fact, we can reroute the paths so that the intersection of P_i and each path in R' is a path. Then we claim that these $40r$ paths in R' together with $P_1, \dots, P_{18r}$ of $\mathbf{P}$ would give rise to an $r \times r$ grid minor. The proof of this claim follows from the proof of (4), so we omit it. But now we have an $r \times r$ grid minor in F.

Similarly, we can prove the case that there is a bridge whose attachments are both in P_1 and P_{18r}. The proof is identical to that of (4), so we leave it to the reader. This completes the proof of Case 1.

Finally, suppose that there are more than $4k$ H''_i's each of which is a 4-vertex cycle. We can apply the whole argument of the previous case to H''_i except that, if H_{7i-3} is a 4-vertex cycle, then $\bigcup_{i=1}^{4} Z'_{i,j}$ could only hit at most r paths of $\mathbf{P}$. So this case is actually easier. Then it is possible to prove that $Z_{j,7i-3} \cup P_j \cup P_{j+1}$ is planar, up to 3-separations, for $j = 1, 2, 3, 4$. So we omit the proof.

This completes the proof of Lemma 22. □

Proof of Theorem 15: We shall follow Diestel et al.'s [27] approach. Actually, the proof here is almost identical to that of Diestel et al.'s.

Set $w = 20^{3k} \cdot 145920k^2 r$. Because $20^{4k} r \geq 7kw \geq (2 \cdot 3k + 2)w$, by Lemma 19, there is a w-mesh of order at least $(2 \cdot 3k + 1)w$. Let $T \subseteq A$ be a tree associated with the premesh (A, B). Hence $X = (A \cap B) \subseteq T$. By Lemma 20, T has at least $3k$ disjoint subtrees each containing at least w vertices of X. Let $A_1, \dots, A_{3k}$ be the vertex sets of these subtrees. Then by the definition of w-mesh, B contains a set $\mathbf{P}_{ij}$ of w disjoint paths between A_i and A_j that have no inner vertices in A.

Let us impose a linear ordering on the index pairs ij by fixing an arbitrary bijection $f : \{ij \mid 1 \leq i \leq 3, 1 \leq j \leq k\}$ to $\{0, \dots, 3k-1\}$.

Let l^* be an integer such that, for all $0 \leq l < l^*$ and all i, j, there exist sets $\mathbf{P}^l_{ij}$ satisfying the following conditions:

1. $\mathbf{P}^l_{ij}$ is a set of disjoint paths from A_i to A_j. Let $H^l_{ij} := \bigcup \mathbf{P}^l_{ij}$, that is, the graph consists of the union of the paths of $\mathbf{P}^l_{ij}$.

2. If $f(ij) < l$, then $\mathbf{P}^l_{ij}$ has exactly one path P_{ij}, and P_{ij} does not meet any paths in $\mathbf{P}^l_{st}$ with $ij \neq st$.

3. If $f(ij) = l$, then $|\mathbf{P}^l_{ij}| = w/20^{2l}$.

4. If $f(ij) > l$, then $|\mathbf{P}^l_{ij}| = w/20^{2l+1}$.

5. If $l = f(pq) < f(ij)$, then for every edge $e \in E(H^l_{ij}) \backslash E(H^l_{pq})$, there are no $w/20^{2l+1}$ disjoint paths from A_i to A_j in the graph $(H^l_{ij} \cup H^l_{pq}) - e$.

We choose l^* as large as possible. If $l^* = 3k-1$, then we are done because there is a $K_{3,k}$-minor. Hence we may assume that $l^* < 3k-1$. We shall first prove that $l^* > 0$. Let $pq = f^-(0)$ and put $\mathbf{P}^0_{pq} := \mathbf{P}_{pq}$. Let $H_{ij} := \bigcup \mathbf{P}_{ij}$ and $F \subseteq E(H_{ij}) \backslash E(H^0_{pq})$ be maximal such that there are still $w/20$ disjoint paths from A_i to A_j in $(H_{ij} \cup H^0_{pq}) - F$. Then, for any $f(ij) > 0$, let $\mathbf{P}^0_{ij}$ be such a set of paths. Then it is easy to see that $\mathbf{P}^0_{ij}$ satisfies the above conditions. This proves that $l^* > 0$.

Having shown that $l^* > 0$, let us now consider $l = l^* - 1$. Thus, Conditions 1–5 above are satisfied for l but cannot be satisfied for $l+1$. Let $l = l^* - 1$. Let $pq = f^-(l)$. We claim that there is no path $P \in \mathbf{P}^l_{pq}$ such that P avoids a set L_{ij} of some $|\mathbf{P}^l_{ij}|/20$ paths in $\mathbf{P}^l_{ij}$ for all ij with $f(ij) > l$. Suppose such a path P exists. Let $st := f^-(l+1)$ and put $\mathbf{P}^{l+1}_{st} := L_{st}$. For $f(ij) > l+1$, let $H_{ij} := \bigcup L_{ij}$, and $F \subseteq E(H_{ij}) \backslash E(H^{l+1}_{st})$ be maximal such that there are still $|\mathbf{P}^l_{ij}|/20^2$ disjoint paths from A_i to A_j in $(H_{ij} \cup H^{l+1}_{st}) - F$. Let $\mathbf{P}^{l+1}_{ij}$ be such a set of paths. If $\mathbf{P}^{l+1}_{pq} := \{P\}$ and $\mathbf{P}^{l+1}_{ij} := \mathbf{P}^l_{ij}$ for $f(ij) < l$, then these would give rise to a family of sets $\mathbf{P}^{l+1}_{ij}$, a contradiction to the maximality of l^*. Thus, for every path $P \in \mathbf{P}^l_{pq}$, P must intersect all but at most $|\mathbf{P}^l_{ij}|/20$ paths in $\mathbf{P}^l_{ij}$ for some $ij > l$. Because $l^* < 3k-1$, there are at least $|\mathbf{P}^l_{pq}|/3k$ paths (letting these paths P'') each of which intersects all but $|\mathbf{P}^l_{ij}|/20$ paths in $\mathbf{P}^l_{ij}$ for some ij. Clearly $|\mathbf{P}^l_{ij}| \geq 20r$ and $|\mathbf{P}^l_{pq}|/3k \geq 48640kr$. By Lemma 22 with $\mathbf{P} := \mathbf{P}^l_{ij}$ and $\mathbf{Q} := P''$, either there is a $K_{3,k}$-minor or there is an $r \times r$ grid minor. Note that Condition 5 gives rise to the minimality in Lemma 22. This completes the proof. □

7.2 Proofs of Decomposition Theorems for $K_{3,4}$-minor-free and K_6^--minor-free Graphs

In this section we prove Theorems 16 and 17. Because the proofs of Theorems 16 and 17 are similar, we will prove them simultaneously.

Proof of Theorems 16 and 17: Suppose G may be expressed as a proper k-sum of A^+ and B^+. Clearly, if G does not contain a minor isomorphic to H, then neither A^+ nor B^+ contains a minor isomorphic to H. So, by considering a minimal counterexample to Theorem 16, it suffices to prove the following. Set $w = 20^{2^{15}}$.

Theorem 23 *Every graph G with no minor isomorphic to $K_{3,4}$ is either*

1. *a proper* 0-, 1-, 2-, *or* 3-*sum of two graphs, or*
2. *planar, or*
3. *has treewidth at most w.*

Similarly, every graph G with no minor isomorphic to K_6^- is either

1. *a proper* 0-, 1-, 2-, *or* 3-*sum of two graphs, or*
2. *planar, or*
3. *has treewidth at most w.*

Suppose that Theorem 16 were not true, and let G be a minimal counterexample. To prove Theorem 16, we need to examine the connectivity property.

Because G cannot be expressed as a proper 0-, 1-, 2-, or 3-sum of two graphs, we have

(1) G has minimum degree at least 3, and no separation (A, B) of order at most 2.

Hence G is 3-connected.

A graph H is called *quasi 4-connected* if it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of G of order three, either $|A| \leq 4$ or $|B| \leq 4$.

The next goal is to prove the following:

(2) G is quasi 4-connected.

Because G cannot be expressed as the proper $3-$ sum of two graphs, this implies that G has no separation of order 3 such that both A and B contain a cycle. Suppose there is a separation (A, B) of order 3 and $|A|, |B| \geq 5$. We may assume that one of A, B, say, A is a forest. Take a component A' in $A - B$. Because A' is a tree, there are at least two vertices of degree exactly 1 in A'. These two vertices have at least two neighbors in $A \cap B$ by (1). But then it is easy to see that there is a cycle in $A' \cup (A \cap B)$. This proves (2).

Because G has treewidth at least w and is quasi 4-connected, it suffices to prove that every quasi 4-connected graph with no minor isomorphic to $K_{3,4}$ and treewidth at least w is planar, and every quasi 4-connected graph with no minor isomorphic to K_6^- and treewidth at least w is planar.

A graph K is a *subdivision* of a graph G if K is obtained from G by replacing its edges by internally disjoint paths with the same end vertices. Let H be a 3-connected planar graph. Then H has a unique planar embedding. In particular, a cycle in H bounds a region in some planar embedding of H if and only if it bounds a region in every planar embedding of H. Such cycles are called *peripheral*. So these cycles may be viewed as "faces" of the embedding. Let u, v be two vertices of H such that no peripheral cycle includes both of them, and let H_1 be the graph obtained from H by adding an edge uv. We say that H_1 is a *jump extension* of H. Let C be a peripheral cycle in H and let u, v, x, y be four distinct vertices of C occurring in this order on C. Let H_2 be the graph obtained from H by adding two edges $\{u, x\}, \{v, y\}$. We say that H_2 is a *cross extension* of H.

The following was shown in [51]. See also [58, Lemma 3.1].

Theorem 24 *Let H' be a quasi 4-connected planar graph with no 3-cycles and let G be a quasi 4-connected nonplanar graph such that G has a subgraph isomorphic to a subdivision of H'. Then G has a subdivision isomorphic to either a jump extension of H' or a cross extension of H'.*

We are ready to finish our proof. If G is planar, then we are done. So we may assume that G is nonplanar. By (2), G is quasi 4-connected. Because G has a 30×30 grid minor, G has a 29×29 grid plus two additional edges joining two corners (not crossing) as a minor. Then, it is easy to see that G has a 15-wall with two additional edges joining two corners (not crossing) as a subdivision. Let us call this graph H (not the subdivision). It is easy to see that H is quasi 4-connected because the minimum degree is at least 3 and there is no separation (A, B) of order at most 3 with $|A|, |B| \geq 5$. Also H contains no 3-cycles. Hence, by Theorem 24, G has a subdivision isomorphic to a jump extension of H or a cross extension of H. Let us call this graph H' (not the subdivision). Then clearly this graph, call it H'', has a 6×6 grid plus two additional crossing edges joining two corners as a minor. Let u, v, x, y be the distinct four corner vertices of H'' occurring in this order along the outer cycle of H'' (minus two crossing edges). Then both $\{u, x\}$ and $\{v, y\}$ are edges. It remains to prove that this graph H'' has a minor isomorphic to $K_{3,4}$ and a minor isomorphic to K_6^-.

Let us first find a $K_{3,4}$-minor in H''. We can specify each vertex of H'' as (i, j) with $1 \leq i \leq 6$ and $1 \leq j \leq 6$ without any confusion. So we may assume that $u = (1, 1)$, $v = (1, 6)$, $x = (6, 6)$, and $y = (6, 1)$. Refer to Figure 3(left).

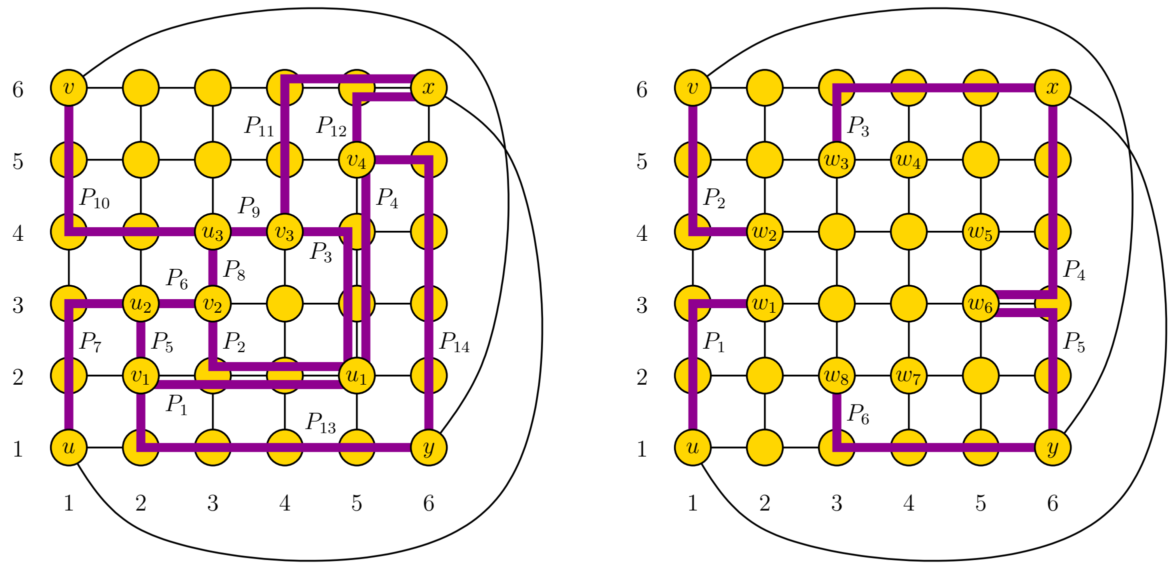

Figure 3: Constructing a $K_{3,4}$ minor (left) and a K_6^- minor (right) from a 6×6 grid with opposite diagonals connected. Paths are described in the body of the proof.

Let $v_1 = (2,2)$, $v_2 = (3,3)$, $v_3 = (4,4)$, and $v_4 = (5,5)$. Also let $u_1 = (5,2)$, $u_2 = (2,3)$, and $u_3 = (3,4)$. We will construct a $K_{3,4}$-minor in such a way that v_i corresponds to a vertex of degree 3 in $K_{3,4}$ for $1 \le i \le 4$ and u_j corresponds to a vertex of degree 4 in $K_{3,4}$ for $1 \le j \le 3$. Because H'' contains a 6×6 grid as a minor and is 3-connected, we can choose the following 14 paths that are pairwise disjoint except at or near common endpoints:

1. P_i connects u_1 and v_i for $i = 1, 2, 3, 4$.
2. P_5 connects u_2 and v_1, P_6 connects u_2 and v_2, and P_7 connects u_2 and u.
3. P_8 connects u_3 and v_2, P_9 connects u_3 and v_3, and P_{10} connects u_3 and v.
4. P_{11} connects x and v_3 and P_{12} connects x and v_4.
5. P_{13} connects y and v_1 and P_{14} connects y and v_4.

By contracting paths P_7 and P_{10}, we get a minor isomorphic to $K_{3,4}$ such that v_i corresponds to a vertex of degree 3 in $K_{3,4}$ for $1 \le i \le 4$ and u_j corresponds to a vertex of degree 4 in $K_{3,4}$ for $1 \le j \le 3$.

Let us find a K_6^--minor in H''; refer to Figure 3(right). Take the 4×4 grid J in the middle of H'', i.e., the distance between the outer face boundary of H'' (minus two crossing edges $\{ux\}, \{vy\}$) and J is exactly 1. By contracting some of the edges of J, we can get an 8-wheel W as a minor. Let w_i be the vertices of the wheel W that are on the outer face boundary of W, appearing in the clockwise order for $i = 1 \ldots, 8$, such that the following paths exist and are disjoint except at or near common endpoints:

1. P_1 connects w_1 and u.
2. P_2 connects w_2 and v.

3. P_3 connects w_3 and x.

4. P_4 connects w_6 and x.

5. P_5 connects w_6 and y.

6. P_6 connects w_8 and y.

7. Furthermore, all six paths $P_1, \ldots, P_6$ are disjoint from the grid J (and hence the wheel W) except at their endpoints.

It is easy to see that such a labeling of w_i is possible. In fact, we can put $w_1 = (2,3)$, $w_2 = (2,4)$, $w_3 = (3,5)$, $w_4 = (4,5)$, $w_5 = (5,4)$, $w_6 = (5,3)$, $w_7 = (4,2)$ and $w_8 = (3,2)$. Then it is easy to see that the wheel W together with paths $P_1, P_2, P_3, P_4, P_5, P_6$ gives rise to a K_6^--minor, by taking w_1, w_2, w_3, w_6, w_8 and the middle vertex of the wheel W as the vertices of K_6^-. Note that the edge w_3w_8 is missing in the corresponding K_6^-.

Hence these two constructions imply that G is planar, and this completes the proof. □

8 Contraction Version of Wagner's Conjecture

Motivated in particular by Kuratowski's Theorem characterizing planar graphs as graphs excluding $K_{3,3}$ and K_5 as minors, Wagner conjectured and Robertson and Seymour proved the following three results:

Theorem 25 (Wagner's Conjecture) [50] *For any infinite sequence $G_0, G_1, G_2, \ldots$ of graphs, there is a pair (i, j) such that $i < j$ and G_i is a minor of G_j.*

Corollary 26 [50] *Any minor-closed graph property[3] is characterized by a finite set of excluded minors.*

Corollary 27 [50, 49] *Every minor-closed graph property can be decided in polynomial time.*

The important question we consider is whether these theorems hold when the notion of "minor" is replaced by "contraction". The motivation for this variation is that many graph properties are closed under contractions but not under minors (i.e., deletions). Examples include the decision problems associated with dominating set, edge dominating set, connected dominating set, diameter, etc.

One necessary difference is how we handle disconnected graphs. For example, the class of all graphs having no edges is characterized by excluding K_2 as a minor. But these graphs are characterized by no finite set of excluded contractions: we must exclude the graph with n vertices and exactly one edge for all $n \geq 2$. Here n represents the number of connected components in the graph. For this reason, we restrict our attention to characterizing graph properties on connected graphs by a finite set of excluded contractions.

One positive result along these lines is about minor-closed properties:

Theorem 28 *Any minor-closed property on connected graphs is characterized by a finite set of excluded contractions.*

[3] A *property* is simply a set of graphs, representing the graphs having the property.

Proof: Let X be the finite set of excluded minors guaranteed by Corollary 26. Let X' be the set of all induced supergraphs of graphs in X, i.e., supergraphs on the same set of vertices. (If desired, we can also discard from X' any graph that can be contracted into another graph in X', to obtain a minimal set X'.) Any graph that has a graph in X' as a contraction also has a graph in X as a minor, so it does not satisfy the property. Similarly, any graph that does not satisfy the property has a graph in X as a minor. If we ignore the edge deletions involved in forming the minor X, and convert any vertex deletion into an incident edge contraction (which exists because the graph stays connected), then we obtain a graph in X' as a contraction. Therefore the finite set X' of excluded contractions characterizes the property. □

For example, we obtain the following contraction version of Kuratowski's Theorem, using the construction of the previous theorem and observing that all other induced supergraphs of $K_{3,3}$ have K_5 as a contraction.

Corollary 29 *Connected planar graphs are characterized by a finite set of excluded contractions, shown in Figure 4.*

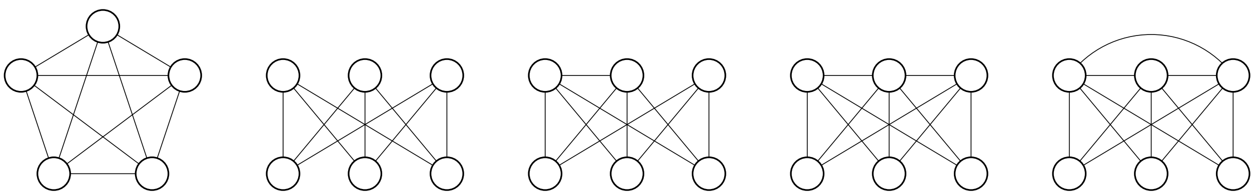

Figure 4: Forbidden contractions for planar graphs.

Another positive result is that Wagner's Conjecture extends to contractions in the special case of trees. This result follows from the normal Wagner's Conjecture because a tree T_1 is a minor of another tree T_2 if and only if T_1 is a contraction of T_2:

Proposition 30 *For any infinite sequence $G_0, G_1, G_2, \ldots$ of trees, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Unfortunately, the contraction version of Wagner's Conjecture does not hold for general graphs:

Theorem 31 *There is an infinite sequence $G_0, G_1, G_2, \ldots$ of connected graphs such that, for every pair (i, j), $i \neq j$, G_i is not a contraction of G_j.*

Proof: Let $G_i = K_{2,i+2}$ for $i = 0, 1, 2, \ldots$. Suppose for contraction that G_i is a contraction of G_j with $i \neq j$. Thus $i < j$. Let v and w denote the two vertices in G_j connected to all other vertices in G_j. The effect of the first contraction in G_j is to remove one vertex, and change one path of length 2 between v and w into a path of length 1 (an edge) between v and w. Thus, after this contraction, v and w form a triangle with every other vertex; thus there are $i+1$ triangles. In order to form G_i, which has no triangles, every triangle must be contracted away. There are two ways to contract away a triangle u, v, w: contract the edge $\{v, w\}$, or contract u into either v or w. If we ever contract the edge $\{v, w\}$, we will be left with a star, which can be contracted only into stars, and never into G_i which has a cycle. Contracting u into either v or w contracts away the triangle u, v, w but no other triangles, and removes one vertex. Thus contracting away all $i+1$ triangles in this way must remove all vertices except v and w, leaving the single edge $\{v, w\}$, which cannot be contracted into G_i which has at least 4 vertices. □

Corollary 32 *There is a contraction-closed property on connected graphs that cannot be characterized by a finite set of excluded contractions.*

Proof: Let $G_0, G_1, G_2, \ldots$ be any infinite sequence of graphs with the property in Theorem 31, such as $G_i = K_{2,i+2}$. Let $\mathcal{G}$ be the set of graphs that cannot be contracted to any G_i. This set is closed under contractions, by definition. We claim that $\mathcal{G}$ cannot be characterized by a finite set of excluded contractions.

Suppose for contraction that there is a finite set $\mathcal{X}$ of graphs such that $\mathcal{G}$ is precisely the set of graphs that cannot be contracted to any graph in $\mathcal{X}$. Because $\mathcal{X}$ is finite, there must be some $G_i \notin \mathcal{X}$. But $G_i \notin \mathcal{G}$, so there must be some $X \in \mathcal{X}$ that is a contraction of G_i. Also, $X \neq G_i$. By the property in Theorem 31, no G_j is a contraction of G_i (for $i \neq j$), so no G_j is a contraction of X. By definition of $\mathcal{G}$, $X \in \mathcal{G}$. But this contradicts that no graph in $\mathcal{X}$ is a contraction of a graph in $\mathcal{G}$. $\square$

The graphs $G_i = K_{2,i+2}$ that form the counterexample of Theorem 31 and Corollary 32 are in some sense tight. Each G_i is a planar graph with faces of degree 4. If all faces are smaller, the contraction version of Wagner's Conjecture holds. A planar graph is *triangulated* if some planar embedding (or equivalently, every planar embedding) is triangulated, i.e., all faces have degree 3. Recall that the triangulated planar graphs are the maximal planar graphs, i.e., planar graphs in which no edges can be added while preserving planarity.

Theorem 33 *For any infinite sequence $G_0, G_1, G_2, \ldots$ of triangulated planar graphs, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Proof: By Theorem 25, there is a pair (i, j) such that $i < j$ and G_i is a minor of G_j. Consider the sequence of edge contractions, edge deletions, and vertex deletions that form G_i from G_j. Because G_j is connected, we can rewrite a vertex deletion as deleting all incident edges except one and then contracting the last edge. Now consider ignoring all edge deletion options from the sequence. The resulting contraction G'_i of G_j must have the same vertex set as G_i, it must include all edges from G_i, and it must be planar. Furthermore, because G_i is maximally planar, G'_i cannot have any additional edges beyond those in G_i. Therefore $G_i = G'_i$, so G_i is in fact a contraction of G_j. $\square$

Another sense in which the counterexample graphs $G_i = K_{2,i+2}$ are tight is that they are 2-connected and are 2-outerplanar, i.e., removing the (four) vertices on the outside face leaves an outerplanar graph (with all vertices on the new outside face). However, the contraction version of Wagner's Conjecture holds for 2-connected (1-)outerplanar graphs:

Theorem 34 *For any infinite sequence $G_0, G_1, G_2, \ldots$ of 2-connected embedded outerplanar graphs, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Proof: We prove a stronger form of the theorem in which every graph has exactly one *marked edge* that is on the boundary face. This marked edge is preserved under contractions in the natural way: after contracting an edge incident to an endpoint v of the marked edge $\{v, w\}$, the new marked edge is the edge between the merged vertex v and the vertex w. The only constraint is that the marked edge itself cannot be contracted.

The marking induces a partial labeling on the graph which constrains what it means for a graph G_i to be a contraction of a graph G_j. The graph resulting from contraction of G_j must be isomorphic to the target graph G_i via an isomorphism that maps the marked edge of G_j to the marked edge of G_i.

Our stronger theorem can be rephrased using the following terminology. A *well-quasi-ordering* $(X, \leq)$ is a set X and a reflexive transitive relation $\leq$ such that, for any infinite sequence $X_0, X_1, X_2, \ldots$ over X, there is a pair (i, j) such that $i < j$ and $X_i \leq X_j$. We say that X is *well-quasi-ordered* over $\leq$. Thus our goal is to prove that the set of 2-connected marked embedded outerplanar graphs is well-quasi-ordered under contraction. Our proof will also consider well-quasi-ordering of other sets under other operations. In particular, we use the following two well-known facts:

Proposition 35 (e.g., [57]) *If $(X, \leq_X)$ and $(Y, \leq_Y)$ are well-quasi-orderings, then $(X \times Y, \leq)$ is a well-quasi-ordering where $(x, y) \leq (x', y')$ if $x \leq_X x'$ and $y \leq_2 y'$.*

Proposition 36 [34] *If $(X, \leq_X)$ is a well-quasi-ordering, then $(X^*, \leq_{X^*})$ is a well-quasi-ordering where X^* is the set of finite sequences over X, and where $(x_1, x_2, \ldots, x_\ell) \leq (x'_1, x'_2, \ldots, x'_k)$ if there is a strictly increasing function $f : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, k\}$ such that $x_i \leq x'_{f(i)}$ for all i.*

Now suppose to the contrary that the theorem is false. Consider a counterexample $G_0, G_1, G_2, \ldots$ such that $\langle |V(G_0)|, |V(G_1)|, |V(G_2)|, \ldots \rangle$ is lexically minimum. That is, choose G_0 with the minimum number of vertices such that there is a counterexample with prefix G_0; then choose G_1 with the minimum number of vertices such that there is a counterexample with prefix G_0, G_1; etc.

For each marked graph G_i, we define a *vector* $A_i = \langle A_{i,1}, A_{i,2}, \ldots, A_{i,d_i} \rangle$ as follows. First observe that no G_i can consist of just a single (marked) edge, even though this graph is 2-connected, because this graph is a contraction of all 2-connected marked outerplanar graphs, in particular G_{i+1}. Thus, because G_i is 2-connected, the marked edge is incident to exactly one finite face F_i. Let d_i be the degree of that face, i.e., the number of edges bounding F_i. The components $A_{i,1}, A_{i,2}, \ldots, A_{i,d_i}$ represent the edges of face F_i in clockwise order starting at the marked edge of G_i. It remains to define how each $A_{i,k}$ "represents" an edge of F_i. An edge e of F_i decomposes the graph into two pieces that overlap at precisely the edge e; $A_{i,j}$ is defined to be the piece to the left of e (oriented clockwise around F_i), i.e., the piece that does not contain F_i. The copy of edge e in $A_{i,j}$ is marked. Note that $A_{i,j}$ may consist only of the edge e (a special 2-connected graph).

For each i and j, we call $A_{i,j}$ a *child* of G_i. The child $A_{i,j}$ has strictly fewer vertices than the parent G_i. Every child $A_{i,j}$ is a contraction of the parent G_i: contract every edge in $A_{i,1}$ except its special edge, contract no edges in $A_{i,j}$, and contract every edge in every other child $A_{i,j'}$. Collectively, $\mathcal{A} = \{A_{i,j} \mid i, j\}$ is the set of all children.

We claim that $\mathcal{A}$ is well-quasi-ordered under contraction. Let $B_0, B_1, B_2, \ldots$ be any sequence over $\mathcal{A}$. For every nonnegative integer i, choose $p(i)$ such that B_i is a child of $G_{p(i)}$. Let ℓ be the nonnegative integer minimizing $p(\ell)$. Because $G_0, G_1, G_2, \ldots$ is a lexically minimum counterexample to the theorem, and because $|V(B_\ell)| < |V(G_{p(\ell)})|$, the theorem holds for $G_0, G_1, \ldots, G_{\ell-1}, B_\ell, B_{\ell+1}, B_{\ell+2}, \ldots$. Thus we obtain a pair (i, j) where $i < j$ and the graph at index i is a contraction of the graph at index j. Because $G_0, G_1, G_2, \ldots$ has no such pair, $j \geq \ell$. Furthermore, $i \geq \ell$ because otherwise G_i would be a contraction of B_j which is a child of, and therefore a contraction of, G_j, a contradiction. Thus B_i is a contraction of B_j, so $B_0, B_1, B_2, \ldots$ is well-quasi-ordered under contraction.

Define a *subvector* of a vector $Q = \langle K_1, K_2, \ldots, K_q \rangle$ of marked graphs to be any vector of the form $\langle K_1, K_{i_2}, K_{i_3}, \ldots, K_{i_\ell} \rangle$, where $1 < i_2 < i_3 < \cdots < i_\ell \leq q$ (and thus $\ell \leq q$). Thus a subvector is like a subsequence, except that the first element must remain intact. We say that a vector $P = \langle J_1, J_2, \ldots, J_p \rangle$ is a *vector-contraction* of a vector Q if there is an subvector $Q' = \langle K_1, K_2, \ldots, K_p \rangle$ of Q such that J_i is a contraction of K_i for all $i = 1, 2, \ldots, p$ (where contraction of marked graphs is defined as above).

Lemma 37 *If a set $\mathcal{A}$ of graphs is well-quasi-ordered under contraction, then the set of vectors over $\mathcal{A}$ is well-quasi-ordered under vector-contraction.*

Proof: We can represent a vector $\langle K_0, K_1, \ldots, K_q\rangle$ as an ordered pair of a graph and a vector, separating off the first graph: $(K_0, \langle K_1, K_2, \ldots, K_q\rangle)$. The well-quasi-ordering of such pairs, involving a graph from $\mathcal{A}$ and a sequence of graphs from $\mathcal{A}$, follows from Propositions 35 and 36. $\Box$

Applying Lemma 37 to the sequence $A_0, A_1, A_2, \ldots$, we obtain a pair (i, j) such that $i < j$ and A_i is a vector-contraction of A_j. Thus there is a sequence of contractions within the elements of A_j and removal of elements of A_j that result in A_i. We can simulate these contractions and removals to show that G_i is a contraction of G_j. The marking of edge of attachment between each child $A_{j,\ell}$ and the face F_j ensures that any contraction within $A_{j,\ell}$ can be mimicked in G_j. For $\ell > 1$, the removal of a child $A_{j,\ell}$ can be simulated in G_j by contracting all edges of $A_{j,\ell}$, resulting in a single vertex. The definition of subvector prevents the first child $A_{j,1}$ from being removed, and thus $A_{j,1}$, which contains the marked edge of G_j, contracts to $A_{i,1}$, which contains the marked edge of G_i. Thus the marked edge of G_j maps to the marked edge of G_i, as required. Therefore, G_i is a contraction of G_j, contradicting that there was a counterexample to the theorem. $\Box$

Corollary 38 *Every contraction-closed graph property of trees, triangulated planar graphs, and/or 2-connected outerplanar graphs is characterized by a finite set of excluded contractions.*

Proof: Follows from Theorems 30, 33, and 34. $\Box$

We can use this result to prove the existence of a polynomial-time algorithm to decide any fixed contraction-closed property for trees and 2-connected outerplanar graphs, using a dynamic program that tests for a fixed graph contraction in a bounded-treewidth graph.

9 Open Problems and Conjectures

One of the main open problems is to close the gap between the best current upper and lower bounds relating treewidth and grid minors. For map graphs, it would be interesting to determine whether our analysis is tight, in particular, whether we can construct an example for which the $O(r^3)$ bound is tight. Such a construction would be very interesting because it would improve the best previous lower bound of $\Omega(r^2 \lg r)$ for general graphs [55]. We make the following stronger claim about general graphs:

Conjecture 39 *For some constant $c > 0$, every graph with treewidth at least cr^3 has an $r \times r$ grid minor. Furthermore, this bound is tight: some graphs have treewidth $\Omega(r^3)$ and no $r \times r$ grid minor.*

This conjecture is consistent with the belief of Robertson, Seymour, and Thomas [55] that the treewidth of general graphs is polynomial in the size of the largest grid minor.

We also conjecture that the contraction version of Wagner's Conjecture holds for k-outerplanar graphs for any fixed k. If this is true, it is particularly interesting that the property holds for k-outerplanar graphs, which have bounded treewidth, but does not work in general for bounded-treewidth graphs (as we have shown in Theorem 31).

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