Metamorphosis of the Cube

Erik Demaine Martin Demaine Anna Lubiw Joseph O'Rourke Irena Pashchenko

Dept. of Computer Science, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
{eddemaine, mldemaine, alubiw}@uwaterloo.ca

Dept. of Computer Science, Smith College
Northampton, MA 01063, USA
{orourke, ipashche}@cs.smith.edu

1 Introduction

The foldings and unfoldings shown in this video illustrate two problems: (1) cut open and unfold a convex polyhedron to a simple planar polygon; and (2) fold and glue a simple planar polygon into a convex polyhedron.

A convex polyhedron can always be unfolded, at least when we allow cuts across the faces of the polyhedron. One such unfolding is the *star unfolding* [2]. When cuts are limited to the edges of the polyhedron, it is not known whether every convex polyhedron has an unfolding [3], although there is software that has never failed to produce an unfolding, e.g., HyperGami [6].

The problem of folding a polygon to form a convex polyhedron is partly answered by a powerful theorem of Aleksandrov. The following section describes this in more detail. After that, we explain the examples that are shown in the video, and discuss some open problems.

2 Aleksandrov's conditions

Aleksandrov's theorem [1] concerns the realization of a "polyhedral metric," assigned to a curved surface, by a convex polyhedron. Here we only consider a version of the theorem that relates to folding a polygon into a convex polyhedron. Suppose we glue each portion of the polygon boundary to another portion of equal length so that the resulting topological surface is homeomorphic to a sphere. Then the theorem states that the surface can be realized by a convex polyhedron provided that the sum of the angles glued together at any point is at most 2π. Furthermore, the realizing polyhedron is unique.

Using Aleksandrov's theorem, Lubiw and O'Rourke [7] give a dynamic programming algorithm that decides in polynomial time whether a polygon can be glued edge-to-edge to satisfy Aleksandrov's conditions. The algorithm can also enumerate all the possible edge-to-edge gluings satisfying Aleksandrov's conditions. This enumeration takes exponential time, but we have shown that there can be an exponential number of valid gluings. We are currently working on extending this algorithm beyond the case where whole edges of the polygon must be glued to other whole edges.

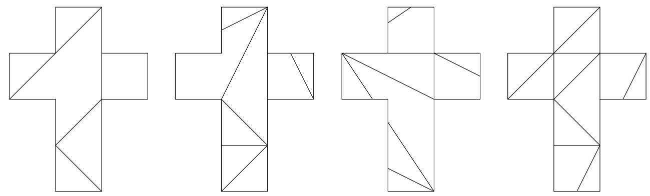

Figure 1: Crease patterns to fold the Latin cross into (from left to right) a doubly covered quadrilateral, a pentahedron, a tetrahedron, and an octahedron.

There is a considerable gap between finding a gluing satisfying Aleksandrov's conditions and constructing the actual convex polyhedron his theorem guarantees. It seems natural to divide the problem into two steps: finding, in the polygonal shape, the fold lines or creases that will form the edges of the polyhedron; and finding the dihedral angles at these edges. A superset of the creases can be found by constructing on the surface of the polyhedron all shortest paths between vertices, that is, points at which the sum of the glued angles is less than 2π. The edges of the polyhedron will be contained in these paths; the remaining portions of the paths will maintain dihedral angles of π.

This leaves the second step, which is an algorithmic version of Cauchy's rigidity theorem: given the combinatorial information about containment of edges and faces of a purported convex polyhedron, and given the geometry of each face, find the actual convex polyhedron formed. Cauchy's rigidity theorem [4] guarantees uniqueness. It is an open problem to find an algorithm for this version of Cauchy's rigidity theorem. An iterative method has been explored [8] that uses springs to relax the edge to their correct lengths, while "pumping air" into the polyhedron to maintain convexity. While this approach has promise, it is only a numerical approximation algorithm.

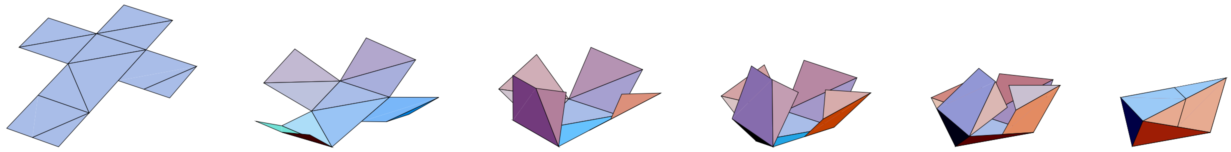

Figure 2: Folding the Latin cross into the octahedron.

3 Examples

In the video we first show two standard unfoldings of the cube, formed by cutting along edges of the cube. The second one is the familiar Latin cross.

Then from the cross, we show all the possible polyhedra than can be formed by folding and gluing edges to edges. The first, a flat doubly covered quadrilateral, is perhaps not so surprising. But the remarkable result is that the cross can be folded into three other nonflat polyhedra: a 5-vertex pentahedron, a 4-vertex tetrahedron, and a 6-vertex octahedron. These gluings were found using the algorithm from [7]. The crease patterns are shown in Fig. 1; a sample folding is shown in Fig. 2.

Next, the video shows a strange unfolding of the cube using cuts across the faces, just to give a taste for the many possibilities.

Finally, we give an unfolding of the cube using face cuts, but resulting in a rectilinear polygon (which we call the "stop-light"), and then show an alternative way to fold this polygon (into a "spaceship"). See Fig. 3.

As mentioned in the previous section, we have no algorithm to find the actual polyhedron formed by a particular gluing of a polygon. Our examples were constructed by hand. The creases were found by making paper models. The dihedral angles at the edges of the polyhedra were found by ad hoc calculations using spherical trigonometry and exploiting the symmetry of the objects. For the octahedron we resorted to measuring the dihedral angles on a cardboard model!

In our folding animations, we rotated the two faces at an edge uniformly from angle π to the correct dihedral angle. We do not know whether this will in general keep the faces from intersecting each other, although it appears to have done so in our examples.

4 Open Questions

We have already mentioned two major open questions above. See [8] for an exposition of these problems.

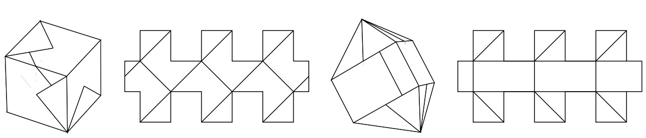

Figure 3: An unusual unfolding of the cube into an orthogonal polygon, and another folding of this polygon.

The progressions shown in the video prompt a question related to dissections:

1. Given two polyhedra, can you unfold one and fold it to form the other?

Frederickson's book on dissections [5] mentions dissecting the surface of polyhedra, but without the restriction that the unfolded surface be connected and simple. In particular he shows an example, due to Theobald, of cutting the surface of a cube and unfolding into two polygons that can then be reglued and folded to form a regular tetrahedron. We transformed the cube into a tetrahedron using only one connected simple piece, but the resulting tetrahedron is not regular. Hence a very specific question:

2. Can a cube be unfolded to a polygon that can then be folded to form a regular tetrahedron?

Another question raised by the surprising unfoldings of the cube is simpler but less well-defined:

3. Characterize the class of simple polygons and non-simple polygons that are unfoldings of a given convex polyhedron, such as the cube.

Acknowledgments. These animations were produced using Mathematica. We thank Rick Mabry for Mathematica advice; and Patrick Gilhuly and Mark Riddell for assistance in producing the video. This work was supported by NSERC and NSF.

References

[1] A. D. Aleksandrov. *Konvexe Polyeder.* Akademie Verlag, 1958.

[2] P. K. Agarwal, B. Aronov, J. O'Rourke, and C. A. Schevon. Star unfolding of a polytope with applications. *SIAM Journal on Computing*, 26(6):1689–1713, Dec. 1997.

[3] H. T. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, 1991.

[4] P. R. Cromwell. *Polyhedra.* Cambridge University Press, 1997.

[5] G. N. Frederickson. *Dissections: Plane and Fancy.* Cambridge University Press, 1997.

[6] B. Hayes. Pleasures of plication. *American Scientist*, Nov.–Dec. 1995.

[7] A. Lubiw and J. O'Rourke. When can a polygon fold to a polytope? Tech. Rep. 048, Smith College, June 1996.

[8] J. O'Rourke. Folding and unfolding in computational geometry. Proc. Japan Conf. Discrete and Computational Geometry, LNCS, Tokyo, Japan, Dec. 1998, to appear.