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Sand drawings and Gaussian graphs*

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Sand drawings form a part of many cultural traditions. Depending on the part of the world in which they occur, such drawings have different names such as *sona*, *kolam*, and *nitus* drawings. In this paper, we show connections between a special class of sand drawings and mathematical objects studied in the disciplines of graph theory and topology called *Gaussian graphs*. Motivated by this connection, we further our study to include analysis of some properties of sand drawings. In particular, we study the number of different drawings, show how to generate them, and show connections to the well-known Traveling Salesman Problem in computer science.

Keywords: Ethnomathematics; Eulerian graphs; Generic planar closed curves

AMS Subject Classification: 01A07; 05C10; 52C99

1. Introduction

Different cultures around the world contain mathematical ideas in one form or another. Not all of these mathematical ideas have developed out of necessity, like counting and calculation: some develop in the context of cultural arts. The study of this mathematical art within and without its cultural context constitutes the field of *ethnomathematics* [2–5].

In this paper, we explore the mathematics and geometry found in a particular kind of visual art that seems to have developed independently in different forms in disparate cultures. In its basic form, the artist draws dots followed by one surrounding continuous loop, which crosses itself repeatedly, in the sand or on a floor sprinkled with powder. Collectively, we refer to these practices as *sand drawings*, though each practicing culture has its own name for them.

For example, women in Tamil Nadu (South India) create geometric designs using rice flower, called *kolam*, at the entrances of their homes [3]. One type of kolam, called *pulli kolam*, consists of first drawing a grid of dots (the *pulli*), and then drawing a continuous closed curve that partitions the planar space into as many bounded regions as there are dots, such that each bounded region contains exactly one dot. Each kolam drawing has a name. Figure 1 illustrates two such drawings.

The *Tshokwe* people of the West Central Bantu area of Africa make similar drawings called *sona*. Here men draw several dots and a continuous closed curve in the sand using their fingers. Drawings that consist of one continuous curve are called *monolinear*; some of these drawings, however, are composed of more than one closed curve. Figure 5 shows two examples of 2-linear drawings [2] (consisting of two curves). In this paper, we consider only monolinear sona drawings. It may also happen in these drawings that some regions contain more than one dot, while other regions are empty [2, 6]. Figure 2 shows two such drawings: the drawing on the left has two empty bounded regions, while the drawing on the right has one bounded region with five dots.

Paulus Gerdes [4, 6] has studied some of the mathematical ideas behind sona drawings. In [6], he divides these drawings into classes based on the dimensions of the grid of dots and their method of construction, and studies monolinearity and symmetry as cultural values. Gerdes also describes rules that transform a 2-linear drawing into a monolinear one, and chain rules that "chain" two monolinear drawings together to form one bigger drawing;

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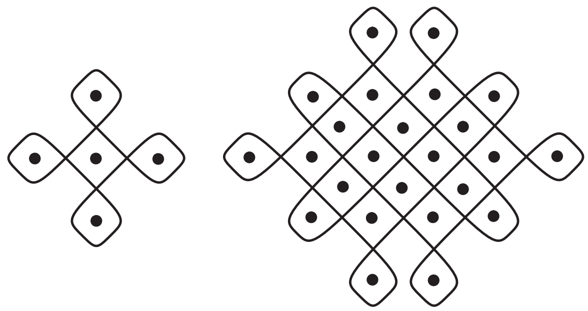


Figure 1. Two examples of *pulli kolam*: the *Anklet of Krishna* (left) and *The Ring* (right).

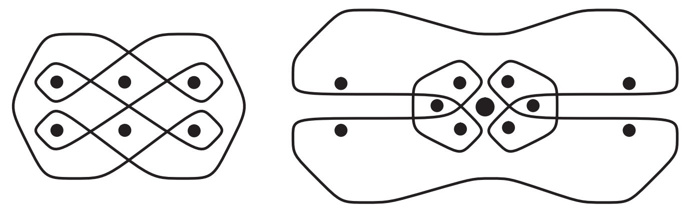


Figure 2. Two examples of *sona*: the *Antelope's Paw* (left) and the *Spider* (right).

moreover, he develops several geometric algorithms for constructing some families of *sona* drawings. One such algorithm uses Euclid's algorithm for computing the greatest common divisor of two natural numbers. It is interesting that the Euclidean algorithm [7] not only generates traditional drawings in visual art, but also traditional rhythms in music [8].

In other traditions, people make drawings on the sand without first placing a grid of dots. One such example is the *nitus* sand drawings by the people of Malekula, an island in the South Pacific that is part of the Republic of Vanuatu. Marcia Ascher [9] has analysed these drawings from the graph theoretic, geometric, and topological points of view.

From a geometric perspective, a sand drawing can be viewed as a self-intersecting closed curve drawn in the two-dimensional plane. The curve must be *generic* (formally, an immersion of the unit circle into the plane) in the sense that the curve is not tangent to itself anywhere and no more than two portions of the curve cross at any point. Arnol′d [10], for example, describes some properties of these curves and shows that they are topological invariants. While his results about such curves were most likely developed independent of sand drawings, there is on the other hand mathematical research inspired by sona drawings. Mirror curves and cycle matrices are two examples, and a recent overview of research inspired by sona can be found in [6].

In this paper, we unveil another connection between sand drawings and mathematics, namely, a concept in graph theory introduced by Gauss in 1830. We explore a variety of basic questions about these "Gaussian graphs" and show how these properties appear in real-life sand drawings. We start in Section 2 by describing the connection to graph theory. Then, in Section 3, we explore how to use this connection to generate all possible different sand drawings, providing a source for new sand drawings. In Section 4, we consider a basic mathematical question about how "regular" a sand drawing can be. Finally, Section 5 shows a connection to the famous Traveling Salesman Problem in computer science.

2. Connections to Gauss and Graph Theory

Imagine applying the following procedure to a sand drawing, as illustrated in figure 3. Draw a disk, called a *vertex*, at each intersection point where the curve meets itself. The curve visits these vertices in some order; we refer to each piece of the curve that connects two consecutive vertices as an *edge* between those two vertices. This structure of vertices and edges is called a *graph*. Graphs generated in this way from sand drawings have a special structure, and we call them *sona graphs* (after the *Tshokwe* people).

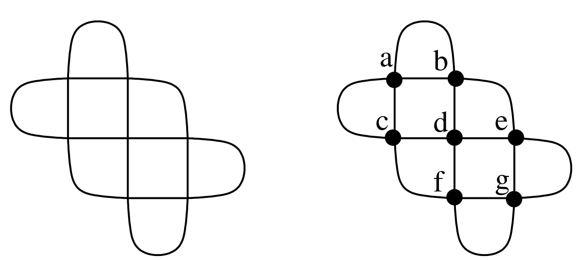


Figure 3. A sona drawing (left) and its corresponding sona graph with eight vertices a through g (right).

The first property of sona graphs is that you can visit the edges one at a time, visit each edge exactly once, and end up where you started. Such graphs are called *Eulerian* after the famous 18th-century Swiss mathematician, Leonhard Euler. Euler showed that a graph has this property precisely when it is both connected (it is possible to walk between any two vertices by traversing edges) and every vertex has an even number of edges touching it. Because a sand drawing does not visit the same vertex more than twice, sona graphs have an even stronger property: each vertex has exactly four edges touching it. Such a graph is called 4-*regular*.

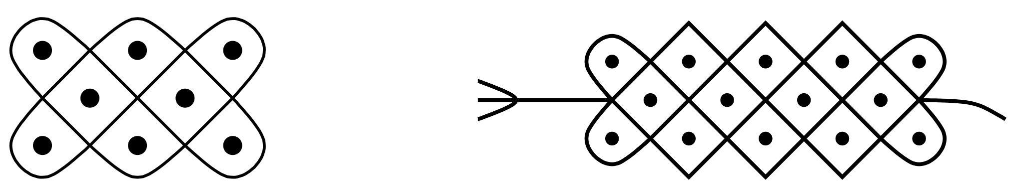
The second property of sona graphs is that, if you traverse the edges by "going straight" at each vertex, then you visit every edge exactly once before returning to your starting point. More precisely, if we label the four edges around a vertex by $1, 2, 3, 4$ in clockwise order, then "going straight" connects edge 1 to edge 3, edge 2 to edge 4, and vice versa. This property is even stronger than being Eulerian; it is called *Gaussian*. Gaussian graphs implicitly go back to an observation made by the famous 19th-century German mathematician, Carl Friedrich Gauss, around 1830 [11]. In his study about the theory of knots, Gauss noted that traversing a generic closed curve in the plane visits an intersection point twice, once in an odd position and once in an even position. His observation was proved by Julius v. Sz. Nagy almost a hundred years later [12]; see also [13, 14]. The formal notion of Gaussian graphs was introduced more recently by Michael Gargano and John Kennedy [14], and then generalized by John Kennedy and Brigitte and Herman Servatius [15]. We essentially follow Gargano and Kennedy's notion, which assumes that the graph is 4-regular as we do.

The regions outlined by the edges of a sand drawing are called *faces*. The number of such faces is of particular importance; we denote the number by n. Note that we do not count the unbounded region outside the curve as a "face".

We can use Euler's Formula—so famous that it even appeared on a stamp of the former German Democratic Republic in 1983—to relate the number of vertices, edges, and faces in a sand drawing. Euler's Formula says that the number of vertices minus the number of edges plus the number of faces is 1 for any connected graph drawn in the plane. Euler's Formula is usually written $V - E + F = 2$; in our case, the 2 becomes a 1 because we do not count the outside region as a face. The number of faces is n. Because every vertex has exactly four edges touching it, and every edge touches exactly two vertices, the number of edges is twice the number of vertices. Plugging this fact into Euler's formula, we conclude that the number of faces minus the number of vertices is 1; in other words, the number of vertices is $n-1$. The number of edges is twice this number, i.e., $2\,n-2$. Therefore, every sona graph has $n - 1$ vertices, $2\,n - 2$ edges, and n faces.

3. All Possible Sand Drawings

The two sona drawings in figure 4 have the same overall structure, but differ in the number of faces. For the Tshokwe people, these two drawings have different meanings and representations. In general, sand drawings having the same structure but different numbers of faces are considered different in cultural practices. This motivates our interest in the number of faces in a sand drawing. On the other hand, two drawings with the same number of faces but different structures are also considered to be different. Therefore, we are also interested in the structure of these drawings. The goal of this section is to characterize all possible sand drawings on a given number n of faces.



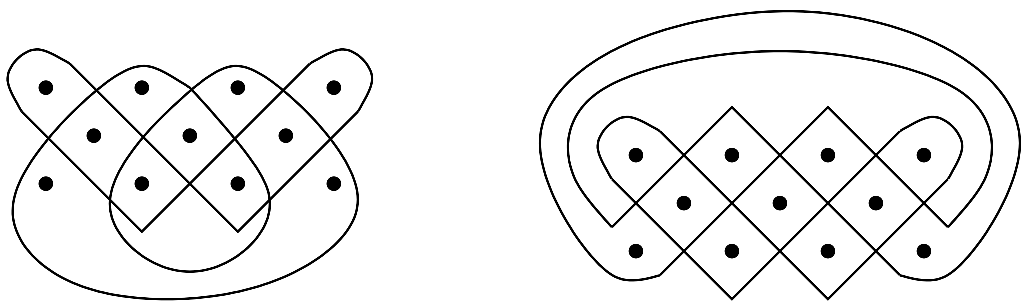
(a) "A small animal that lives in a tree hole and pierces the intestines" [2]

(b) "A kind of rat" [6]

Figure 4. Ignoring the "head" and the "tail" of the rat, these two drawings are examples of sona drawings having the same overall pattern but different meanings.

Before attempting to answer this question, we first need to specify when we consider two sand drawings to be different in structure. In the previous section, we established that a sona graph is a 4-regular Gaussian graph. But the graph structure—which vertices are connected to which other vertices by edges—is not necessarily the whole story. When we draw a sona graph in the plane, the structure of the drawing depends on how we embed its vertices and edges. A *planar embedding* of a graph is a placement of the vertices as points in the plane and a drawing of the edges as curves that intersect each other only at common endpoints. A sona graph together with a planar embedding is called a *sona map*. In general, two sand drawings are considered to be different when they correspond to different sona maps.

But there are often exceptions to this rule. The example in figure 5 given by Ascher [2] shows two different sona drawings (different in the sense of sona maps), but which have the same underlying graph structure (they define the same sona graph). For the Tshokwe people, these two drawings have the same name and tell the same story. Thus, sona graphs are also of interest.



(a) (b)

Figure 5. In this sona drawing, the rightmost and leftmost dots on the bottom row "represent Sa Chituku and his wife, Na Chituku; the other dots are their neighbors. Sa Chitiku [sic] builds barriers that isolate his wife from the neighbors so that she will attend to cooking instead of visiting" [2]. (Note that this example actually consists of two separate loops.)

If we want to count the number of possible sand drawings on a given number of faces, there are two different objects we can count: sona graphs and sona maps. Each sona graph has one or more associated sona maps, so in general there are more maps than graphs. How many sona maps and how many sona graphs are there on n faces? Exact solutions to these questions seem difficult, but we have established upper and lower bounds, both of which are singly exponential in n (precisely, between 2^{cn} and $2^{c'n}$ for constants c and c') [1].

One approach to characterizing sand drawings is to design algorithms for generating all of them. In particular, generation makes it easy to count them all. Kennedy et al. [15] show how to recursively construct all possible sona maps by applying a sequence of *edge splices* to the figure eight graph. Roughly, as shown in figure 6, an edge-splice operation takes two edges of a common face and pinches these edges together at a newly created vertex, splitting the face in two. By trying all possible edge splices from all possible sona maps on $n-1$ faces, we obtain all possible sona maps on n faces. One challenge is to discard duplicate copies of sona maps on n faces, but this too can be done efficiently by testing pairs of maps for a match, guessing two matching features and traversing the maps from there. Thus we obtain an efficient algorithm for generating all sona maps on a given number of faces.

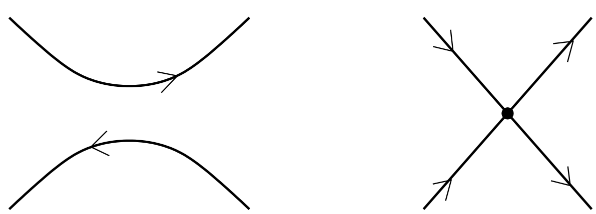


Figure 6. An edge-splice operation takes two edges (left) and pinches them together at a new vertex (right).

To obtain all sona graphs on a given number of faces, the best algorithm we are aware of is to take all possible sona maps on the same number of faces, and search for pairs of maps that have the same underlying graph. This search problem is called *graph isomorphism*, a classic problem for which no one knows an efficient algorithm for general graphs. We can use this approach to generate all possible sona graphs on a given number of faces, but the algorithm requires exponential time for each graph, so it becomes impractical for large numbers of faces.

We have developed a software program that generates all sona graphs and sona maps on n faces, for small n. More precisely, the program generates all distinct sona graphs on n faces, and all distinct sona maps on n faces, incrementally for $n = 2, 3, 4, \ldots$, using the algorithms described above. Table 1 shows the computed number of

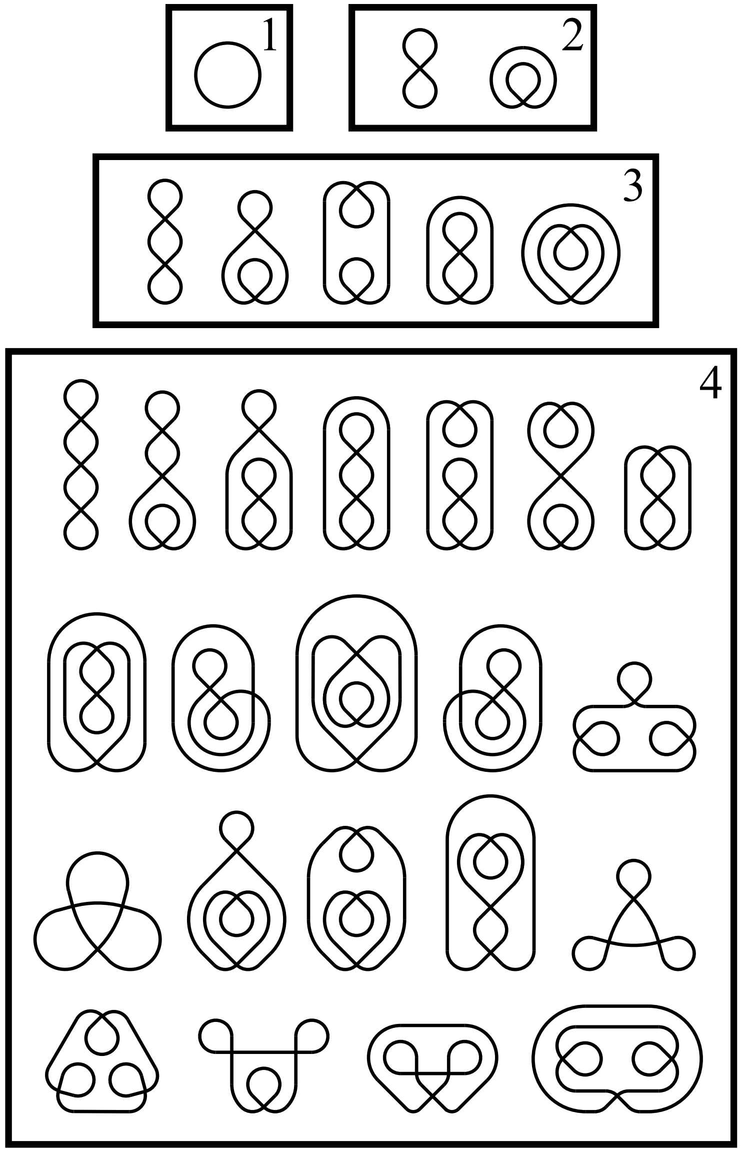


Figure 7. All sona maps on n faces for n between 1 and 4.

sona graphs and sona maps on n faces for n between 1 and 9. After $n = 9$, the (rather inefficient) program would take too long to run.

The program also draws a polygonal planar embedding of each sona map, where each edge is represented by a chain of up to three line segments, by triangulating and applying a theorem of Tutte [16]. Unfortunately, these drawings make poor use of area and are barely visible without zooming in extremely close. We manually redrew each sona map for n between 1 and 4 using a combination of circular arcs and straight lines, joined at common

n faces	sona graphs	sona maps
$n = 1$	1	1
$n = 2$	1	2
$n = 3$	1	5
$n = 4$	3	21
$n = 5$	5	102
$n = 6$	13	639
$n = 7$	38	4,492
$n = 8$	133	34,032
$n = 9$	534	272,252

Table 1. The number of sona graphs and sona maps on n faces for small n.

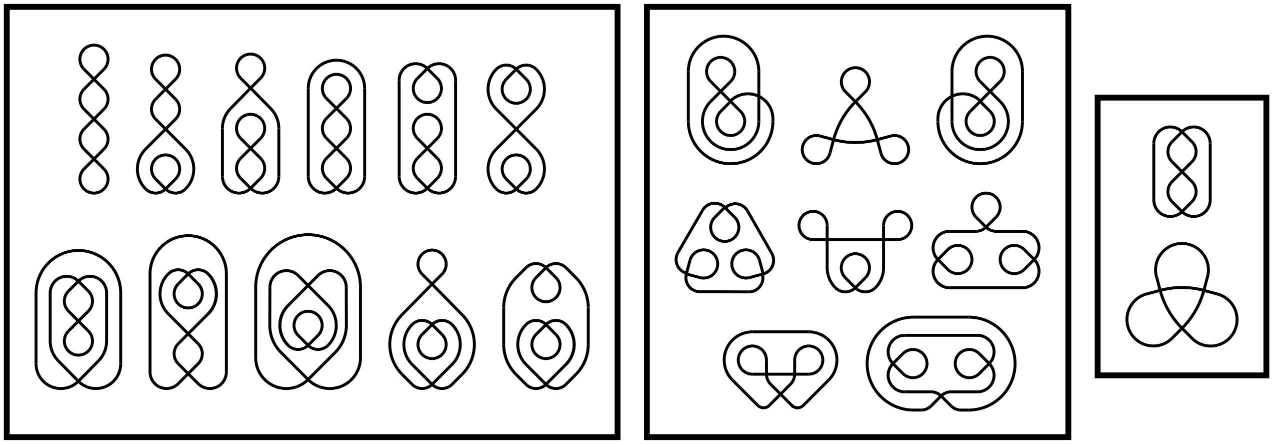


Figure 8. The sona maps on 4 faces grouped by their underlying sona graphs.

tangents, so that the resulting curve always crosses itself at right angles, and to illustrate all symmetries in the map. Figure 7 shows these drawings of all sona maps for n between 1 and 4. To better understand the difference between sona maps and sona graphs, it helps to arrange the 21 sona maps on 4 faces into three groups such that the maps of each group correspond to the same underlying sona graph. Figure 8 shows the three groups for $n = 4$.

We distinguish sona maps from their reflections, but the only sona maps in figure 7 that lack reflectional symmetry are the second and fourth maps in the second row of $n = 4$, which resemble a treble clef. The sona map in between these two maps, consisting of a loop nested within double edges, resembles a rose; it is perhaps the most complicated sona map in the sense of requiring the most area to draw.

4. Regularity in Sand Drawings

Examining many of the sand drawings shown in [4, 6] reveals an initially surprising fact: most drawings have a large number of faces bounded by the same number of edges. We refer to this regular structure as the *regularity* of a sand drawing. In this section, we investigate this regularity through the "face degree sequence" of sona maps. The *degree* of a face is the number of edges bounding the face. The *face degree sequence* is the sequence of the degrees of all faces, sorted by degree. Figure 9 shows an example of two sona maps with the same face degree sequence.

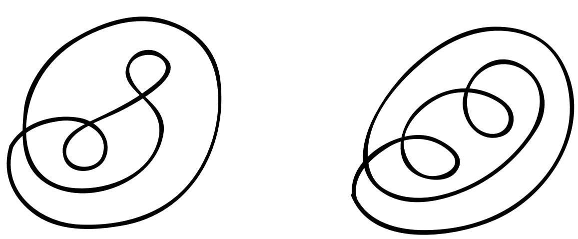


Figure 9. Two different sona maps with the same face degree sequence: $1, 1, 4, 5$.

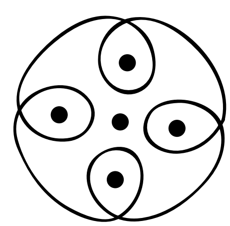


Figure 10. A sona map in which all but one face has degree 1.

To understand what values are possible in the face degree sequence, we consider the *average* face degree. If we sum up all the face degrees, we end up counting most edges exactly twice, because most edges bound exactly two faces each. The exception is the edges bounding the entire drawing, which get counted only once because they bound only one face. Our computation using Euler's Formula in Section 2 showed that the number of edges is $2\,n - 2$, where n is the number of faces. Thus, the sum of all face degrees must be smaller than $4\,n - 4$, smaller by 1 for each edge bounding the entire drawing. We obtain the average degree by dividing by the number n of faces, resulting in a value slightly smaller than 4.

What does this average face degree tell us about regularity? First, the majority of faces must have degrees between 1 and 6: if more than half the faces had degree at least 7, then the average face degree would be more than $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 7 = 4$. Second, every sand drawing must have a significant regularity: some degree between 1 and 4 must appear at least one tenth of the time; some degree between 1 and 5 must appear at least two fifteenths of the time; some degree between 1 and 6 must appear at least one seventh of the time; and so on. Third, if most of the faces (all but a subconstant fraction) have the same degree, that degree must be between 1 and 4.

Inspired by these mathematical features of regularity, we study sand drawings for any fixed number of faces n in which most of the faces have the same degree.

(i) **Degree 1:** There is a sona map on n faces with $n-1$ faces of degree exactly 1. Figure 10 shows the construction. Here, each face contains exactly one dot and the curve loops around each of the n dots in turn. For the last face, however, it just circles around the dot and connects to the starting point. Interestingly, such a construction appears as a real-world pulli kolam drawing; see figure 1 (left).

(ii) **Degree 2:** There is a sona map on n faces with $n - 2$ faces of degree exactly 2, and with two faces of degree exactly 1. Figure 11 shows the construction, which takes a loop of string and twists the two strands (reversing their order). Interestingly, this construction appears as an element in real-world sona drawings; see figure 12.

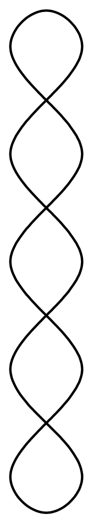


Figure 11. A sona map in which all but two faces have degree exactly 2.

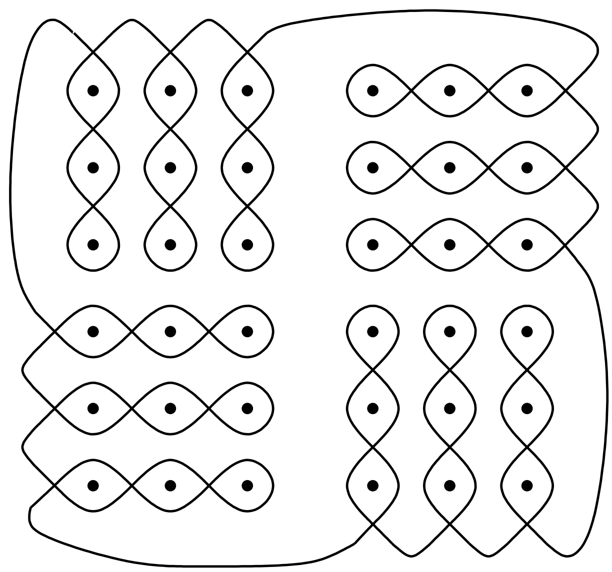


Figure 12. "Men-lions that, stealthily, plan their intrigues" [4].

(iii) **Degree 3:** There is a sona map on n faces where all but a small constant number of faces have degree exactly 3. Specifically, if n is one more than a multiple of 3, there is a sona map on n faces with $n - 3$ faces of degree exactly 3. Figure 13 shows the construction, which starts from the construction from figure 11 and, just before closing the loop, continues the drawing by crossing and dividing each face into two, and then returns to the original point to close the loop. Curiously, we have not seen this construction in real-world sand drawings.

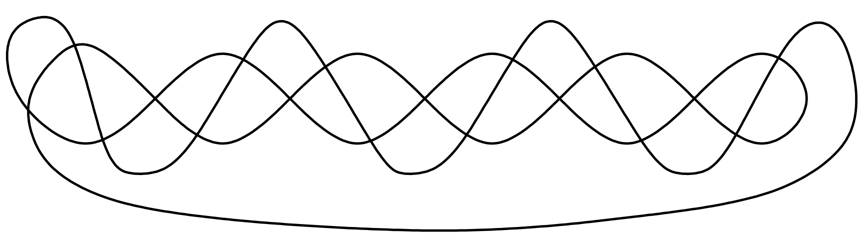


Figure 13. A sona map in which all but three faces have degree exactly 3.

(iv) **Degree 4:** There is a sona map on n faces where almost all faces have degree exactly 4. A first example is a sona map that forms a square grid, with faces of degree exactly 4, except for some anomalies on the boundary of the grid required to make the graph Gaussian. The number of faces with degree other than 4 is less than the number of boundary edges, $4\sqrt{n}$, which is negligible compared to the number of faces of degree 4. Interestingly, such constructions appear in real-world sona drawings; see figure 4(b). A similar grid-like construction appears in the pulli kolam of figure 1.

A second example is inspired by the sona drawing in figure 14. In that drawing, nine of the seventeen faces have degree exactly 4; the other eight have degree 3. For any n, we can construct a similar sona map, having one dot in each face, by placing the dots in a plus sign, with roughly $n/4$ dots in each of the four spokes from the center. Starting from the central dot (the "spider"), we make clockwise turns and loop around the dots, creating vertices around the four extreme dots and the four dots surrounding the spider. The remaining $n - 8$ faces have degree 4, if n is one more than a multiple of four. In general, we obtain a small constant number of faces with degrees other than 4.

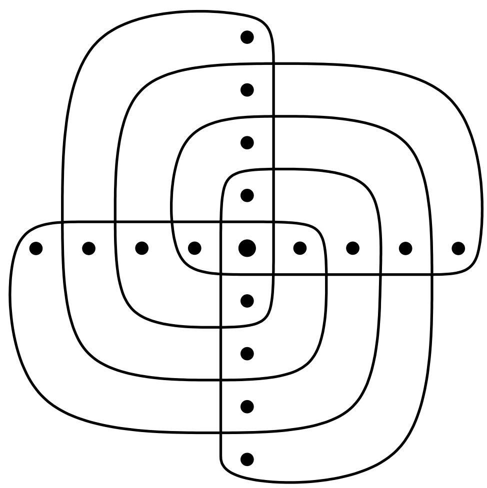


Figure 14. "A spider in its cobweb" [4]. The four outermost faces and the four surrounding the central face have degree 3; all other faces have degree 4.

5. Connections to the Traveling Salesman Problem

What is the "best" sand drawing for a given placement of dots if each face of the drawing must contain exactly one of the dots? This question is difficult to answer because it requires defining aesthetics: smoothness, symmetry, and story all play a role. But from a purely geometric perspective, a natural objective is to find a sona map of minimum possible total length having exactly one of the given dots in each face. We call this the *minimum-length* sona map.

Although the minimum-length sona map is probably not interesting artistically, it has an interesting connection to the famous Traveling Salesman Problem (TSP). In TSP, we are given a collection of points, and the goal is to find a tour of minimum possible total length that visits every point and returns to where we started. It is computationally intractable (NP-hard) to find the minimum-length TSP tour.

For a given set of dots, there is the minimum-length sona map and there is the minimum-length TSP tour. How do these two lengths compare? Given a TSP tour, there is a natural way to build a sona map: loop around each dot in turn with very small loops, connecting the loops together according to the TSP tour, as in figure 10. We do not loop around the last dot, but we do visit it to ensure that it is enclosed by the high-degree face. The resulting sona map is just slightly longer than the TSP tour. Is this the best possible?

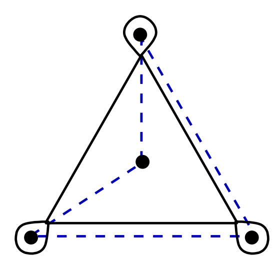


Figure 15. The TSP tour (dashed lines) is slightly longer than the sona map whose faces encircle the given dots.

Figure 15 shows an example where the minimum-length TSP tour is longer than the minimum-length sona drawing. The example has four dots, three dots forming an equilateral triangle with side length 1 and the fourth dot in the center of the triangle . The minimum-length TSP tour has length $2 + 2\sqrt{3}/3 > 3.15$, while the minimum-length sona map visits only the corners of the triangle, at a cost just slightly larger than 3. So the minimum lengths can differ by about 5%. In fact, Damian et al. [17] show that the lengths are never too different: the minimum-length TSP tour of the given dots is less than a factor of $(\pi + 2)/\pi \approx 1.64$ longer than the minimum-length sona map. So the two lengths are closely related, separated by small constant factors.

6. Conclusion

It is exciting to discover the connections between art forms found in different cultures around the world and formal mathematical concepts that were developed in the West. The connection between sand drawings and Gaussian graphs is one such example. This connection opens up many new directions of pursuit that might help us understand the underlying mathematical structure of these drawings, raising many interesting open problems for mathematicians to work on.

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