On Wrapping Spheres and Cubes with Rectangular Paper

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1 Introduction

Given a flat unit-area rectangle of paper with dimensions $x \times 1/x$ (with $x \leq 1$), what is the largest sphere or cube fully wrappable? This problem has been extensively studied and has vital applications in mass production, where a more efficient wrapping of a candy or other small object could translate to substantial savings. We summarize and generalize existing techniques for upper and lower bounds, and provide new techniques for improved bounds.

Catalano-Johnson and Loeb [4] provide an optimal cube wrapping with a unit square. They also derive cube upper bounds by exploiting antipodal points. Akiyama et al. [3] provide foldings for narrow rectangles and 6 different fixed-ratio rectangles by skewing the paper over an edge-unfolding of a cube. [2] demonstrates an infinite set of perfectly efficient tetrahedron foldings, which we translate to the sphere. Demaine et al. [5] illustrates optimally folding a square into a sphere, and additionally translate [4]'s antipodal points argument to the sphere. Finally, from folklore comes a clean $1/\sqrt{7} \times \sqrt{7}$ wrapping of a cube.

Throughout the paper, the notion of wrapping is identical to that of [5], i.e. a contractive noncrossing mapping from the paper to Euclidian 3-space. An S-cube is a cube with side length S; an S-tetrahedron is a tetrahedron with side length S; an $x \times y$-stadium is the Minkowski sum of a length-x line segment and a y-radius disk.

Our results are collated in Figures 1 and 2.

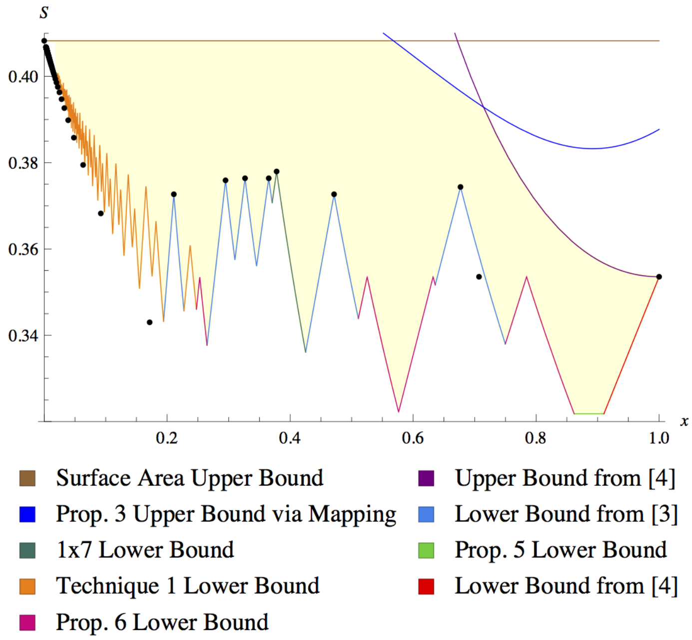

Figure 1: Summary of results on the cube. We simplify presentation by only showing wrappings when they are the best-known. The horizontal axis is the shorter dimension in the $x \times 1/x$ rectangular paper. The vertical axis is the side length of the cube being wrapped. The light yellow regions indicate gaps where upper and lower bounds do not coincide.

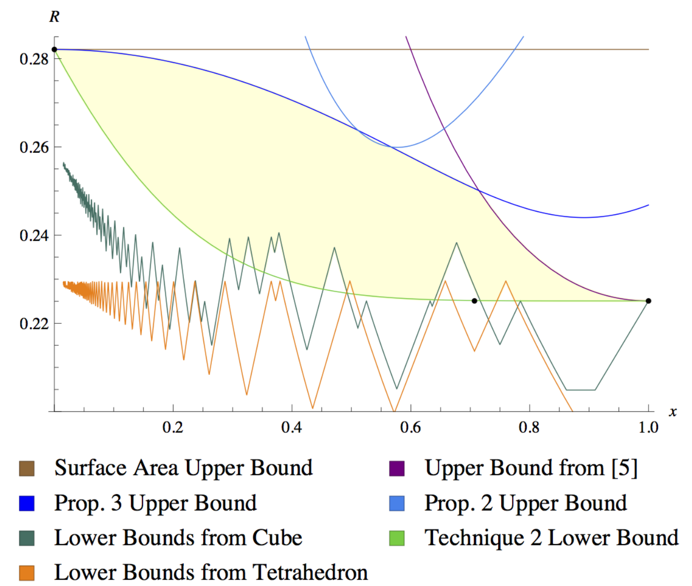

Figure 2: Summary of results on the sphere. The horizontal axis is the shorter dimension in the $x \times 1/x$ rectangular paper. The vertical axis is the radius of the sphere being wrapped. The light yellow regions indicate gaps where upper and lower bounds do not coincide.

2 Contractive Mappings

To translate wrappings between surfaces, we present contractive mappings from cubes and tetrahedra to spheres. By composing a paper-to-X folding with an X-to-Y contractive mapping, we can translate lower bounds on X to Y. Similarly, upper bounds on Y can be mapped back to X.

Proposition 1. *S-cubes and S-tetrahedra can be contractively mapped to spheres with radii at most $2S/\pi$ and $S/(2\sqrt{3}\cos^{-1} 3^{-.5})$, respectively.*

3 Upper Bounds on Spheres

Because 3 congruent circles wrapping a sphere must each be able to cover a hemisphere, we obtain upper bounds for paper of any aspect ratio:

Proposition 2. *An $x \times 1/x$ flat paper rectangle can wrap an radius-R sphere only if $R \leq \sqrt{x^2 + (3x)^{-2}}/\pi$.*

This is achieved by cutting our paper into 3 rectangles and then expanding each into a circle and bounding the resulting foldings. Similar techniques fail to produce new upper bounds on the cube.

Our second approach improves upon the surface area upper bound for any aspect ratio of paper. We do this by inscribing a stadium and, using Proposition 3, discounting the sphere surface area occupiable by the paper.

Proposition 3. *An $x \times y$-stadium of paper mapped onto a radius-R sphere occupies no more surface area than $2R(\pi R - \pi R \cos \frac{x}{2R} + y \sin \frac{x}{2R})$.*

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4 Lower Bounds on Cubes and Spheres

Given a lower bound for one aspect ratio, the following translates it to other dimensions of paper.

Proposition 4. *Given a construction folding a $x \times 1/x$ rectangle into an S-cube, then there exists foldings of $x' \times 1/x'$ rectangles into $f(x')$-cubes where $f(x') = \min(sx'/x,\ sx/x')$.*

This is achieved by shrinking a wrapping to a predetermined aspect ratio, transforming any discrete set of wrappings into a continuous lower bound.

The following two techniques each produces an infinite series of bounds with no concise mathematical formulation.

Technique 1. *The 6 skewed wrappings in [3] can be extended to an infinite series by spiralling around 4 faces of a cube with perfect efficiency and covering the top and bottom with gadgets (Figure 3).*

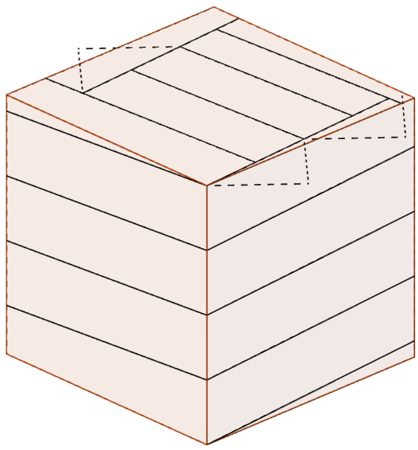

Figure 3: Spiral-wrapping a cube.

Technique 2. *Spiralling paper around a sphere, as visualized in Figure 4, produces an infinite series of wrappings, many of which are the best known for their aspect ratios.*

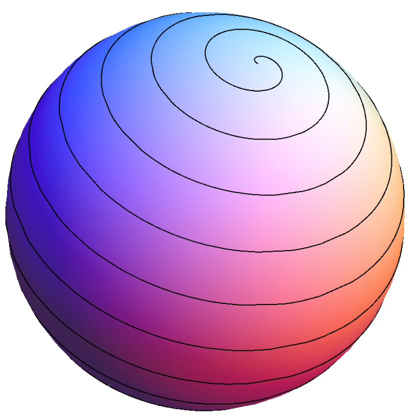

Figure 4: Spiral-wrapping a sphere.

Hinged dissection and the rectangle transformations in [1] provide a powerful technique to rearrange a rectangle of one aspect ratio into one of another while preserving a wrapping. We find a simple folding of a fixed ratio rectangle such that the left and right edges are mapped to one path, and the top and bottom each fold down to a point. Next, this folding is dissected into another rectangle of arbitrary aspect ratio such that all cuts made will be glued back together in the final wrapping. This can be seen in Figure 5.

Proposition 5. *A $1/(2\sqrt{1+\sqrt{2}})$-cube can be wrapped by any unit-area rectangle.*

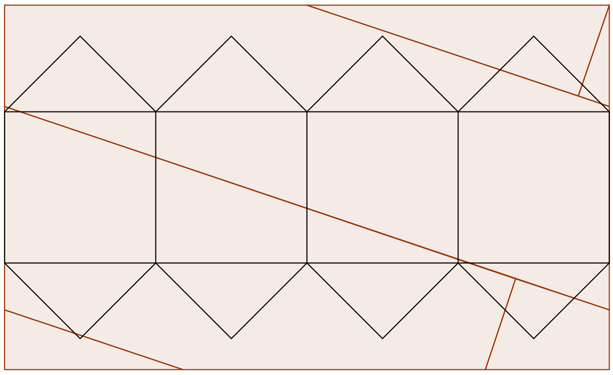

Figure 5: The black lines show a basic cube wrapping. The tan lines show the "cut and paste" method to transfer the wrapping to another rectangle.

The tetrahedral wrappings in [2] inspire a method of wrapping cubes. Here we use a technique similar to that employed in Proposition 5, except that the initial cube wrapping is more efficient as the top and bottom each fold to a path instead of a point. With more careful alignment of the dissection, we get a set of discrete, efficient wrappings:

Proposition 6. *A $1/(2\sqrt{2})$-cube can be wrapped by rectangles with $x = 2\sqrt{2}/\sqrt{n^2+4}$ for integer $n \geq 2$.*

5 Conclusions

Our results are summarized in Figures 1 and 2. Prior lower bounds are shown as points because they applied only to fixed aspect ratios; our results cover rectangles in any shape. Our stadium upper bound is a substantial improvement on the sphere, to such an extent that it translates to the cube, yielding the first new cube upper bound in over a decade.

Much work remains, but we are hopeful that the techniques outlined above will assist in making the bounds match.

References

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