COMMON UNFOLDINGS OF POLYOMINOES AND POLYCUBES

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Abstract Common planar unfoldings of various classes of polyominoes and polycubes are studied. A common unfolding of all tree-like tetracubes was exhaustively discovered by Knuth and Miller; we show here that no common unfolding exists for all tree-like pentacubes.

1 Introduction

Polyominoes A *polyomino* or k*-omino* is an orthogonal polygon made from k squares of unit size arranged with coincident sides. The coincident sides of two adjacent squares either overlap or are joined together. A polyomino is *tree-like* if its dual graph is a tree (each node a square, arcs corresponding to joined sides) and is *incision-free* if it does not have overlapping sides; overlapping corners are allowed.

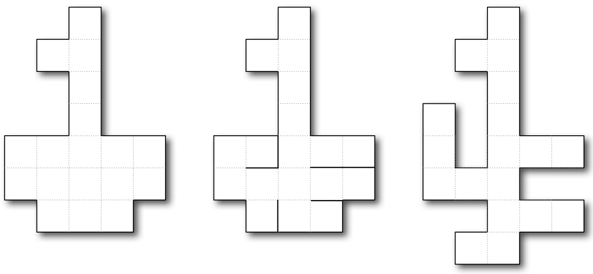
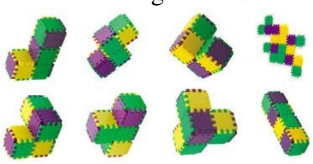


Figure 1. Three octakaidecominoes: non-tree-like (left), tree-like (middle), and incision-free tree-like (right).

Polycubes The 3D analogues of polyominoes are polycubes which are orthogonal polyhedra made from unit cubes arranged with coincident faces which either overlap or are joined together. A polycube is *tree-like* if its dual graph is a tree and is *incision-free* if it does not have overlapping faces; corner overlap is allowed. A *planar* polycube has its dual graph lying on one plane, i.e., it is only one cube thick. A *chiral* polycube has a non-superimposable-under-rigid-motion mirror image. *Chiral twins*, chiral polycubes that are the mirror image of each other, are considered to be distinct.

1.1 Related work Cubigami7 [2] is a commercialized puzzle developed by Donald Knuth and George Miller which solves this problem for $k \leq 4$. Miller posed the question, and Knuth exhaustively tried all unfoldings of all tree-like tetracubes. Nothing is known for $k > 5$.

Cubigami7:



For boxes, i.e., convex polycubes, Biedl *et al.* [1] found two polyomino unfoldings that folds into two distinct boxes. Mitani and Uehara [3] have shown that there are an infinite number of such polyomino unfoldings. The problem of finding a polyomino unfolding that folds into more than two distinct boxes is still open.

Related to this topic is the problem of finding an edge unfolding of some classes of orthogonal polyhedra; see [4] for a survey.

2 Unfolding polyominoes

The perimeter of a k-omino forms a closed chain of $2k+2$ unit length segments. An *unfolding* of a k-omino is defined as follows: open the chain, cut the k-omino into smaller polyominoid pieces partially attached to the chain and straighten the chain. This definition is motivated by the problem of unfolding tree-like planar polycubes (see §3.2). A *valid* unfolding does not overlap and every piece lies on the same side of the chain. A trivial unfolding exists for all polyominoes: open the chain, keep the polyomino identical and leave it connected to an arbitrary segment of the chain. We ask: Is there a common unfolding of all tree-like k-ominoes?

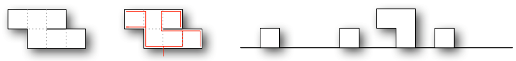


Figure 2. A possible unfolding of a tree-like 6-omino.

2.1 Path-like polyominoes A polyomino is *path-like* if its dual graph is a path. Let $\{c_1, c_2, \ldots, c_{2k+2}\}$ be the ordered sequence of segments composing the unfolded chain C of a k-omino where c_1 is the left most segment. A *caterpillar* unfolding of a k-omino is a chain where one unit size square is attached to each segment $c_{2i+\lfloor 2i/k+1 \rfloor}$ for $1 < i \leq k$ (see Fig. 3). Note that the caterpillar unfolding cannot fold into all path-like polyominoes but we show that it can fold into any *non-spiraling* path-like polyomino. The difference between the number of left and right turns in a non-spiraling path-like polyomino does not exceed 3 if k is even and does not exceed 4 otherwise.

We have proven the following (proofs omitted):

Theorem 1 *For any positive value of* k*, the caterpillar unfolding is a common unfolding of all one-sided non-spiraling path-like* k*-ominoes.*

Corollary 2 *For any odd positive integer* k*, there exists an unfolding that folds to at least* $C_{(k-1)/2}$ *one-sided* k*-ominoes (or at least* $C_{(k-1)/2}/2$ *free* k*-ominoes),*

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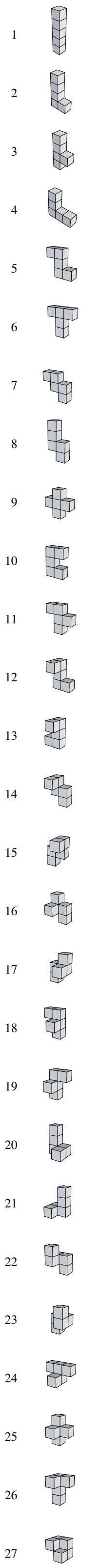


Figure 4 All 27 incision-free pentacubes

where C_n is the nth Catalan number.

The notion of a caterpillar unfolding and the results of Theorem 1 and Corollary 2 can be extended to path-like planar polycubes.

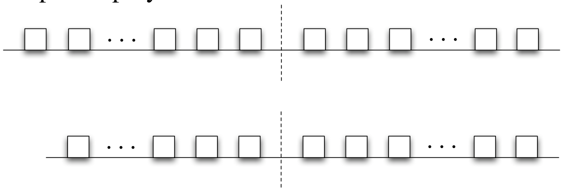


Figure 3. Caterpillar unfolding of k-ominoes for k even (top) or odd (bottom).

2.2 Tree-like polyminoes The following lemma has been verified experimentally:

Proposition 3 For $0 < k \leq 6$, there exists a common unfolding of all tree-like incision-free k-ominoes. There is no common unfolding of all tree-like incision-free heptominoes.

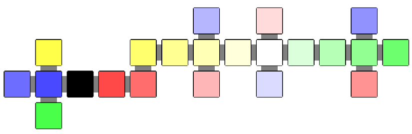
The question remains open for $k \geq 8$.

3 Unfolding polycubes

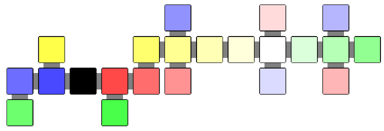
A polyomino unfolding of a polycube is a cutting of a collection of edges on the surface of a polycube so that the surface may be flattened into a polyomino without overlapping faces. We study the problem of finding common polyomino unfoldings of all tree-like k-cubes. A common unfolding of several polyominoes is *one-sided* if all polyominoes can be folded with the same side of the unfolding on the exterior. A mutual unfolding which is not one-sided is *two-sided*.

3.1 Unfolding pentacubes The Cubigami puzzle is a common unfolding of all incision-free tetracubes. We investigate the problem of finding common polyomino unfoldings of all incision-free pentacubes (Fig. 4). The results described in this section are experimental. Using a ULB supercomputer, 1,099,511,627,776 possible unfoldings of the 5-tube (Fig. 4, #1) were generated, and for each generated unfolding it was determined which of the 27 incision-free pentacubes could be folded to. This yielded the following discoveries:

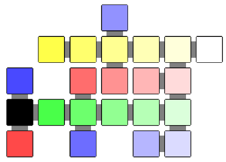
- There is no common two-sided polyomino unfolding of all tree-like incision-free pentacubes (even when chiral twins are considered identical).
- There are several two-sided polyomino unfoldings that can fold to 23 pentacubes (one is pictured below). None fold to 24.



- There are 27 one-sided polyomino unfoldings that can fold to 22 pentacubes. None can fold to more than 22.
- There is an unique two-sided polyomino unfolding of all tree-like incision-free non-planar pentacubes (Fig 4 #12-27). This unfolding, shown below, can fold to a total of 22 pentacubes.



- There are 492 common one-sided polyomino unfoldings of all tree-like incision-free planar pentacubes (Fig 4 #1-11). None of them can fold to a non-planar pentacube. One of them is:



- There is no common two-sided polyomino unfolding of all path-like incision-free pentacubes.
- The smallest subset of pentacubes that have no common polyomino unfolding has a size of at most 4. For example pentacubes (numbered in Fig. 4) 1, 9, 27 and any of 6, 7, 12, 13, 24 or 26 have no common polyomino unfolding.

It remains open if there is there some l such that for all $k \geq l$, there are always two incision-free k-polyominoes without a common unfolding?

3.2 Tree-like planar polycubes

Proposition 4 *For $0 < k \leq 6$, there exists several common one-sided polyomino unfoldings of all tree-like incision-free planar k-cubes.*

The question is open for $k > 6$.

3.3 Non-planar polycubes Nothing is known about finding a common polyomino unfolding of all non-planar k-cubes for $k > 5$. It is open if all tree-like incision-free polycubes have at least one polyomino unfolding.

References

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[4] J. O'Rourke. Unfolding orthogonal polyhedra. *In J. E. Goodman, J. Pach, and R. Pollack, editors, Surveys on Discrete and Computational Geometry — Twenty Years later, Contemporary Mathematics*, 453:307–318, 2008.