Deflating The Pentagon

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Abstract

In this paper we consider deflations (inverse pocket flips) of quadrilaterals and pentagons. We characterize infinitely deflatable quadrilaterals by proving necessity of previously obtained sufficient conditions. Then we show that every pentagon can be deflated after finitely many deflations, and that any infinite deflation sequence of a pentagon results from deflating an induced quadrilateral on four of the vertices.

1 Introduction

A *deflation* of a simple planar polygon is the operation of reflecting a subchain of the polygon through the line connecting its endpoints such that (1) the line intersects the polygon only at those two polygon vertices, (2) the resulting polygon is simple (does not self-intersect), and (3) the reflected subchain lies inside the hull of the resulting polygon. A polygon is *deflated* if it does not admit any deflations, i.e., every pair of polygon vertices either defines a line intersecting the polygon elsewhere or results in a nonsimple polygon after reflection.

Deflation is the inverse operation of pocket flipping. Given a nonconvex simple planar polygon, a *pocket* is a maximal connected region exterior to the polygon and interior to its convex hull. Such a pocket is bounded by one edge of the convex hull of the polygon, called the *pocket lid*, and a subchain of the polygon, called the *pocket subchain*. A *pocket flip* (or simply *flip*) is the operation of reflecting the pocket subchain through the line extending the pocket lid. The result is a new, simple polygon of larger area with the same edge lengths as the original polygon. A convex polygon has no pocket and hence does not admit a flip.

In 1935, Erdős conjectured that every nonconvex polygon convexifies after a finite number of flips [3]. Four years later, Nagy [1] claimed a proof of Erdős's conjecture. Recently, Demaine et al. [2] uncovered a flaw in Nagy's argument, as well as other claimed proofs, but fortunately correct proofs remain.

In the same spirit of finite flips, Wegner conjectured in 1993 that any polygon becomes deflated after a finite number of deflations [6]. Eight years later, Fevens et al. [4] disproved Wegner's conjecture by demonstrating a family of quadrilaterals that admit an infinite number of deflations. They left an open problem of characterizing which polygons deflate infinitely.

In this paper, we show that the family of quadrilaterals described in [4] are the only polygons with four sides that admit infinitely many deflations, thus characterizing infinitely deflatable quadrilaterals. We also show that any such quadrilateral flattens in the limit. We use this characterization of infinitely deflating quadrilaterals to understand deflations of pentagons. Specifically, we show that every pentagon admitting an infinite number of deflations induces an infinitely deflatable quadrilateral on four of its vertices. Then we show our main result: every pentagon can be deflated after finitely many (well-chosen) deflations.

2 Definitions and Notations

Let $P = \langle v_0, v_1, \ldots, v_{n-1}\rangle$ be a polygon together with a clockwise ordering of its vertices. Let $P^k = \langle v_0^k, v_1^k, \ldots, v_{n-1}^k\rangle$ denote the polygon after k arbitrary deflations, and P^* denote the limit of P^k, when it exists, having vertices v_i^*. Thus, the initial polygon $P = P^0$. The *turn angle* of a vertex v_i is the signed angle $\theta \in (-180°, 180°]$ between the two vectors $v_i - v_{i-1}$ and $v_i - v_{i+1}$. A vertex of a polygon is *flat* if the angle between its incident edges is 180°, i.e., forming a turn angle of 0°. A *flat polygon* is a polygon with all its vertices collinear. A *hairpin* vertex v_i is a vertex whose incident edges overlap each other, i.e., forming a turn angle of 180°. A polygon vertex is *sharpened* when its absolute turn angle decreases.

3 Deflation in General

In this section, we prove general properties about deflation in arbitrary simple polygons. Our first few lemmata are fairly straightforward, while the last lemma is quite intricate and central to our later arguments.

Lemma 1 *Deflation only sharpens angles.*

This result follows from an analogous result for pocket flips, which only flatten angles (see, e.g., [5]).

Corollary 2 *Any n-gon with no flat vertices will continue to have no flat vertices after deflation, even in an accumulation point.*

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Lemma 3 *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, the absolute turn angle $|\tau_i|$ at any vertex v_i has a (unique) limit $|\tau_i^*|$.*

Corollary 4 *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, v_i^* is a hairpin vertex in some accumulation point P^* if and only if v_i^* is a hairpin vertex in all accumulation points P^*.*

Lemma 5 *Any n-gon with n odd and no flat vertices cannot flatten in an accumulation point of an infinite deflation sequence.*

Lemma 6 *For any infinite deflation sequence $P^0, P^1, P^2, \ldots$, there is a subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) that is the pocket chain of infinitely many deflations.*

We conclude this section with a challenging lemma showing that infinitely deflating pockets flatten:

Lemma 7 *Assume $P = P^0$ has no flat vertices. If P^* is an accumulation point of the infinite deflation sequence $P^0, P^1, P^2, \ldots$, and subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) is the pocket chain of infinitely many deflations, then $v_i^*, v_{i+1}^*, \ldots, v_j^*$ are collinear and $v_{i+1}^*, \ldots, v_{j-1}^*$ are hairpin vertices. Furthermore, if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_j^*, then v_j^* is a hairpin vertex; and if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_i^*, then v_i^* is a hairpin vertex. In particular, if $j - i = 2$, then either v_i^* or v_j^* is a hairpin vertex.*

Proof. Because $P^0 \supseteq P^1 \supseteq P^2 \supseteq \cdots$, we have $\text{hull}(P^0) \supseteq \text{hull}(P^1) \supseteq \text{hull}(P^2) \supseteq \cdots$, and in particular $\text{area}(\text{hull}(P^0)) \geq \text{area}(\text{hull}(P^1)) \geq \text{area}(\text{hull}(P^2)) \geq \cdots \geq 0$. Thus, $\sum_{t=1}^{\infty} [\text{area}(\text{hull}(P^t)) - \text{area}(\text{hull}(P^{t-1}))] \leq \text{area}(\text{hull}(P^0))$, so $\text{area}(\text{hull}(P^t)) - \text{area}(\text{hull}(P^{t-1})) \to 0$ as $t \to \infty$. Hence, for any $\varepsilon > 0$, there is a time T_ε such that, for all $t \geq T_\varepsilon$, $\text{area}(\text{hull}(P^t)) - \text{area}(\text{hull}(P^{t-1})) \leq \varepsilon$. As a consequence, for all $t \geq T_\varepsilon$, $\text{hull}(P^{t-1}) \subseteq \text{hull}(P^t) \oplus D_{\varepsilon/\ell}$ where $\oplus$ denotes Minkowski sum, D_x denotes a disk of radius x, and ℓ is the length of the longest edge in P, which is a lower bound on the perimeter of $\text{hull}(P^t)$.

Let $t_1, t_2, \ldots$ denote the infinite subsequence of deflations that use $v_i, v_{i+1}, \ldots, v_j$ as the pocket subchain, where P^{t_r} is the polygon immediately after the rth deflation of the pocket chain $v_i, v_{i+1}, \ldots, v_j$. Consider any vertex v_k with $i < k < j$. If $t_r \geq T_\varepsilon$, then $v_k^{t_r - 1} \in \text{hull}(P^{t_r}) \oplus D_{\varepsilon/\ell}$. Also, $v_k^{t_r-1}$ is in the halfplane H_r exterior to the line of support of P^{t_r} through $v_i^{t_r}$ and $v_j^{t_r}$. Now, the region $(\text{hull}(P^{t_r}) \oplus D_{\varepsilon/\ell}) \cap H_r$ converges to a subset of the line $\ell_{i,j}^{t_r}$ through $v_i^{t_r}$ and $v_j^{t_r}$ as $\varepsilon \to 0$ while keeping $t_r \geq T_\varepsilon$. Thus, for any accumulation point P^*, v_k^* is collinear with v_i^* and v_j^*, for all $i < k < j$. In other words, $v_{i+1}^*, \ldots, v_{j-1}^*$ lie on the line $\ell_{i,j}^*$ through v_i^* and v_j^*. By Corollary 2, $v_{i+1}^*, \ldots, v_{j-1}^*$ are not flat, so they must be hairpins.

By Lemma 3, the absolute turn angle $|\tau_j|$ of vertex v_j has a limit $|\tau_j^*|$. If $|\tau_j^*| > 0$ (i.e., v_j^* is not a hairpin in all limit points P^*), then by Lemma 1, $|\tau_j^t| \geq |\tau_j^*| > 0$. For sufficiently large t, v_{j-1}^t approaches the line $\ell_{i,j}^t$. To form the absolute turn angle $|\tau_j^t| \geq |\tau_j^*| > 0$ at v_j, v_{j+1}^t must eventually be bounded away from the line $\ell_{i,j}^t$: after some time T, the minimum of the two angles between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ must be bounded below by some $\alpha > 0$. Now suppose that some $v_k^{t_r-1}$ were to extend beyond $v_j^{t_r-1}$ in the projection onto the line $\ell_{i,j}^{t_r-1}$ for some $t_r - 1 > T$. As illustrated in Figure 1, for the deflation of the chain $v_i^{t_r-1}, v_{i+1}^{t_r-1}, \ldots, v_j^{t_r-1}$ to not cause the next polygon P^{t_r} to self-intersect, the minimum of the two angles between $v_j^{t_r-1} v_k^{t_r-1}$ and $\ell_{i,j}^{t_r-1}$ must also be at least α. See Figure 1.

But this is impossible for sufficiently large t, because v_k^t accumulates on the line $\ell_{i,j}^t$. Hence, in fact, v_k^t must not extend beyond v_j^t in the $\ell_{i,j}^t$ projection for sufficiently large t. In other words, when v_j^* is not a hairpin, each v_k^* must not extend beyond v_j^* on the line $\ell_{i,j}^*$. A symmetric argument handles the case when v_i^* is not a hairpin.

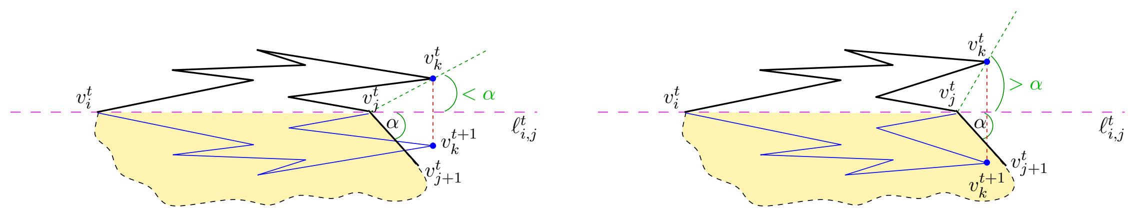
Finally, suppose that $j - i = 2$. In this case, because $v_{i+1}^* = v_{j-1}^*$ is a hairpin, it must extend beyond one of its neighbors, v_i^* or v_j^*. By the argument above, in the first case, v_i^* must be a hairpin, and in the second case, v_j^* must be a hairpin. Thus, as desired, either v_i^* or v_j^* must be a hairpin. □

4 Deflating Quadrilaterals

Lemma 8 *Any accumulation point of an infinite deflation sequence of a quadrilateral is flat and has no flat vertices.*

Proof. First we argue that all quadrilaterals $P^1, P^2, \ldots$ (excluding the initial quadrilateral P^0) have no flat vertices. Because deflations are the inverse of pocket flips, and pocket flips do not exist for convex polygons, deflation always results in a nonconvex polygon. Thus all quadrilaterals P^t with $t > 0$ must be nonconvex. Hence no P^t with $t > 0$ can have a flat vertex, because then it would lie along an edge of the triangle of the other three vertices, making the quadrilateral convex. By Corollary 2, there are also no flat vertices in any accumulation point P^*.

By Lemma 6, there is a subchain $v_i, v_{i+1}, \ldots, v_j$, where $j - i \geq 2$, that is the pocket chain of infinitely many deflations. In fact, $j - i$ must equal 2, because reflecting a longer (4-vertex) pocket chain would not change the polygon. Applying Lemma 7 to $P^1, P^2, \ldots$ (with no flat vertices), for any accumulation point P^*, v_{i+1}^* is a hairpin and either v_i^* or $v_j^* = v_{i+2}^*$ is a



(a) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is less than α, hence in the next deflation step the chain $v_i^t \dots v_j^t$ will intersect the polygon.

(b) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is greater than α, so the polygon will not self-intersect in the next deflation step.

Figure 1: Because v_j^t is not a hairpin, the minimum angle α between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ is strictly positive. If any vertex v_k^t of the chain $v_i^t, v_{i+1}^t, \dots, v_j^t$ extends beyond v_j^t, then the minimum angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ must be at least α for the polygon P^{t+1} not to self-intersect in the next deflation step. The dotted curve represents the rest of the polygon chain and the shaded area is the interior of the polygon below line $\ell_{i,j}^t$.

hairpin. Hairpin v_{i+1}^* implies that v_i^*, v_{i+1}^*, v_{i+2}^* are collinear, while hairpin v_i^* or v_{i+2}^* implies that the remaining vertex $v_{i+3}^* = v_{i-1}^*$ lie on that same line. Therefore, any accumulation point P^* is flat. □

Combining the flattening property of Lemma 8 with the previous necessary conditions of Fevens et al. [4], we obtain a complete characterization of infinitely deflating quadrilaterals:

Theorem 9 *A quadrilateral with side lengths $\ell_1, \ell_2, \ell_3, \ell_4$ is infinitely deflatable if and only if*

1. *opposite edges sum equally, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$; and*

2. *adjacent edges differ, i.e., $\ell_1 \neq \ell_2 \neq \ell_3 \neq \ell_4 \neq \ell_1$.*

Furthermore, every such infinitely deflatable quadrilateral deflates infinitely independent of the choice of deflation sequence.

Proof. Fevens et al. [4] proved that every quadrilateral satisfying the two conditions on its edge lengths are infinitely deflatable, no matter which deflation sequence we make. Thus the two conditions are sufficient for infinite deflation.

To see that the first condition is necessary, we use Lemma 8. Because deflation preserves edge lengths, so do accumulation points of an infinite deflation sequence, so the flat limit configuration from Lemma 8 is a flat configuration of the edge lengths $\ell_1, \ell_2, \ell_3, \ell_4$. By a suitable rotation, we may arrange that the flat configuration lies along the x axis. By Lemma 8, no vertex is flat, so every vertex must be a hairpin. Thus, during a traversal of the polygon boundary, the edges alternate between going left ℓ_i and going right ℓ_i. At the end of the traversal, we must end up where we started. Therefore, $\pm(\ell_1 - \ell_2 + \ell_3 - \ell_4) = 0$, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$.

To see that the second condition is necessary, suppose for contradiction that $\ell_1 = \ell_2$ (the other contrary cases are symmetric). By the first condition, $\ell_1 + \ell_3 = \ell_1 + \ell_4$, so $\ell_3 = \ell_4$. Thus, the polygon is a kite, having two pairs of adjacent equal sides. (Also, all four sides might be equal.) Every kite has a chord that is a line of reflectional symmetry. No kite can deflate along this line, because such a deflation would cause edges to overlap with their reflections. If a kite is convex, it may deflate along its other chord, but then it becomes nonconvex, so it can be deflated only along its line of reflectional symmetry. Therefore, a kite can be deflated at most once, so any infinitely deflatable quadrilateral must have $\ell_1 \neq \ell_2$ and symmetrically $\ell_1 \neq \ell_2 \neq \ell_3 \neq \ell_4 \neq \ell_1$. □

5 Deflating Pentagons

Theorem 10 *There is a pentagon with a flat vertex that deflates infinitely for all deflation sequences, exactly like the quadrilateral on the nonflat vertices.*

Proof. See Figure 2. We start with an infinitely deflating quadrilateral $\langle v_0, v_1, v_2, v_3, v_4 \rangle$ according to Theorem 9, and add a flat vertex v_5 along the edge $v_4 v_0$. As long as we never deflate along a line passing through the flat vertex v_4, the deflations act exactly like the quadrilateral, and thus continue infinitely no matter which deflation sequence we choose. To achieve this property, we set the length of segment $v_3 v_0$ to 1, with v_4 at the midpoint; we set the lengths of edges $v_0 v_1$ and $v_2 v_3$ to 2/3; and we set the length of edge $v_1 v_2$ to 1/3. Then we deflate the quadrilateral until the vertices are close to being hairpins that v_4 cannot see the nonadjacent convex vertex and the line through v_4 and the reflex vertex intersects the pentagon at another point. Thus no line of deflation passes through v_4, so we maintain infinite deflation as in the induced quadrilateral. □

Finally we show that any infinitely deflating pentagon induces an infinitely deflating quadrilateral.

Theorem 11 *Every pentagon with no flat vertices is finitely deflatable.*

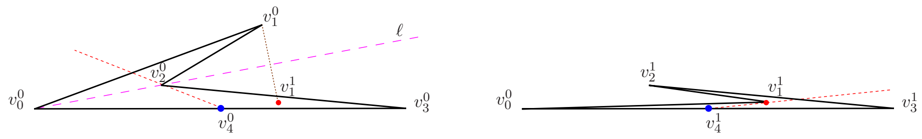


Figure 2: An infinitely deflatable pentagon that induces an infinitely deflatable quadrilateral (left) and its configuration after the first deflation (right).

Proof. Let P be a pentagon with no flat vertices, and assume for the sake of contradiction that P deflates infinitely. Consider any accumulation point P^* of an infinite deflation sequence $P^0, P^1, P^2, \ldots$. By Lemma 6, there is an infinitely deflating pocket chain, say $v_0, v_1, \ldots, v_j$, where $j \geq 2$. By Lemma 7, $v_1^*, \ldots, v_{j-1}^*$ are hairpin vertices. Because the pentagon has only five vertices, $j \leq 4$. In fact, $j \leq 3$: if $j = 4$, this pocket chain would encompass all five vertices, making P^* collinear, which contradicts Lemma 5. If $j = 3$, then v_1^* and v_2^* are hairpins. If $j = 2$, then by Lemma 7, either v_0^* or v_2^* must be a hairpin; suppose by symmetry that it is v_2^*. Thus, in this case, again v_1^* and v_2^* are hairpins. Hence, in all cases, v_1^* and v_2^* are hairpins, so $v_0^*, v_1^*, v_2^*, v_3^*$ are collinear, while by Lemma 5, v_4^* must be off this line.

By Lemma 7, an infinitely flipping pocket chain requires at least one hairpin vertex. Thus, the only possible infinitely flipping pocket chains are v_0, v_1, v_2; v_1, v_2, v_3; and v_0, v_1, v_2, v_3. Let T denote the time after which only these chains flip. Thus, after time T, v_0, v_3, v_4 stop moving, so in particular, v_4's angle and the length of the edge v_0v_3 take on their final values. Therefore, after time T, the vertices v_0, v_1, v_2, v_3 induce a quadrilateral that deflates infinitely.

Because $v_0^*, v_1^*, v_2^*, v_3^*$ are collinear and v_4^* is off this line, neither v_0^* nor v_3^* can be hairpins. By Lemma 7, v_1^* and v_2^* must lie along the segment $v_0^*v_3^*$. By Theorem 9, no two adjacent edges of the quadrilateral have the same length, so in fact v_1^* and v_2^* must be strictly interior to the segment $v_0^*v_3^*$. For sufficiently large t, $v_0^t, v_1^t, v_2^t, v_3^t$ are arbitrarily close to collinear, and v_1^t and v_2^t project strictly interior to the line segment $v_0^tv_3^t$. As a consequence, for sufficiently large t, we can deflate the chain $v_0^t, v_1^t, v_2^t, v_3^t$ along the line through v_0^t and v_3^t into the triangle $v_0^tv_3^tv_4^t$. But then the convex hull of P^{t+1} is $v_0^{t+1}v_3^{t+1}v_4^{t+1}$, which is fixed, so no further deflations are possible, resulting in a finite deflation sequence. □

6 Larger Polygons and Open Problems

It is easy to construct n-gons with $n \geq 6$ that deflate infinitely, no matter which deflation sequence we choose. See Figure 3 for the idea of the construction. We can add any number of spikes to obtain n-gons with $n \geq 6$ and even. For n odd, we can shave off the tip of one of the spikes. Thus, $n = 5$ is the only value for which every n-gon with no flat vertices can be finitely deflated.

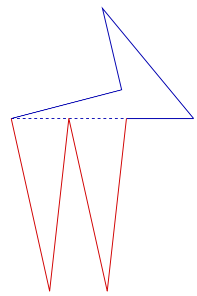


Figure 3: An infinitely deflating octagon constructed by adding long spikes to an infinitely deflating quadrilateral.

None of the infinitely deflating polygons of Figure 3 are particularly satisfying because their accumulation points are not flat. Are there any n-gons, $n > 4$, that have no flat vertices and always deflate infinitely to flat accumulation points? In an unpublished manuscript (2004), Fevens et al. show a family of infinitely deflating hexagons with no flat vertices that flatten in the limit.

Does every infinite deflation sequence have a (unique) limit? Our proofs would likely simplify if we knew there was only one accumulation point.

Is there an efficient algorithm to determine whether a given polygon P has an infinite deflation sequence? What about detecting whether all deflation sequences are infinite? Even given a (succinctly encoded) infinite sequence of deflations, can we efficiently determine whether the sequence is valid, i.e., whether it avoids self-intersection?

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