Narrow Misère Dots-and-Boxes

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Abstract

We study misère Dots-and-Boxes, where the goal is to minimize score, for narrow boards. In particular, we characterize the winner for $1 \times n$ boards with an explicit winning strategy for the first player with a score of $\lfloor (n-1)/3 \rfloor$. We also give preliminary results for $2 \times n$ and for Swedish $1 \times n$ (where the boundary is initially drawn).

1 Introduction

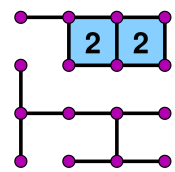

Figure 1: In the middle of a typical Dots-and-Boxes game. Based on [BCG03, Fig. 1].

Recall the classic children's game *Dots-and-Boxes* [BCG03]. We start with an $m \times n$ square grid of dots. Players alternate drawing individual edges of the grid. If a player completes a box of the grid, s/he gets a point and must draw another edge; this process can repeat several times within a single turn. The game ends when all edges have been drawn, i.e., when all mn boxes have been completed. In normal Dots-and-Boxes, the player to receive the most points wins. In misère Dots-and-Boxes, the player to receive the fewest points wins. A draw (tie) occurs when mn is even and the players complete the same number of boxes.

Normal Dots-and-Boxes endgames are known to be NP-hard; see [DH]. In addition, no winning strategies are known when m or n is sufficiently large. To our knowledge, even the $1 \times n$ case is open for arbitrary n. On the other hand, misère Dots-and-Boxes may be easier to analyze.

In Section 2, we give a winning strategy for the first player in $1 \times n$ misère Dots-and-Boxes that guarantees a score of at most $\lfloor (n-1)/3 \rfloor$ boxes, which is a win by roughly $n/6$. The essence of the strategy is to avoid any parity switching of who leads the game, which we show is possible, unlike general boards. In Sections 3 and 4, we give preliminary results for the $2 \times n$ game and for the Swedish $1 \times n$ game (where the boundary is initially drawn).

Terminology. See Figure 2. A *boundary* edge is an edge of the bounding rectangle. An *interior* edge is any nonboundary edge.

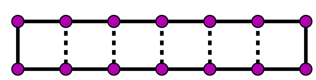

Figure 2: Boundary edges are solid; interior edges are dashed.

2 Misère $1 \times n$

In a $1 \times n$ board, there are $n-1$ interior edges; the remaining $2n+2$ edges are boundary. We distinguish the leftmost and rightmost boxes as *end* boxes.

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Theorem 1 *For all $n \geq 1$, misère $1 \times n$ Dots-and-Boxes is a first-player win.*

Proof: First we describe Player 1's strategy, which divides the game into two phases. In Phase I, some interior edges remain untaken, and Player 1 always takes such an edge. The initial choice of interior edge is any not incident to an end box, if there is one, and otherwise an arbitrary interior edge. We ignore any boxes that Player 2 takes, and instead focus on the last edge played. If Player 2 takes an interior edge, Player 1 takes another arbitrary interior edge. If Player 2 takes a boundary edge, Player 1 takes one of the two incident interior edges, if one of them is untaken, and otherwise an arbitrary interior edge. This rule may cause Player 1 to take a box, in which case Player 1 takes another, arbitrary interior edge (if any exist). In Phase II, when all interior edges are depleted, Player 1 takes any boundary edge that does not complete a box; we will show that such an edge always exists.

We show that no edge can ever complete two boxes simultaneously. During Phase I, Player 1 goes first and takes only interior edges, so the number of taken interior edges is always at least the number of taken boundary edges. Further we claim that, within each nonboundary box except possibly the one in which Player 2 just played, the number of taken interior edges is always at least the number of taken boundary edges. The claim is trivially true before either player has played in the box. If Player 1 plays in the box, the claim certainly remains true. Whenever Player 2 plays in the box, Player 1's next move will be to play in the box, unless both interior edges have already been taken; in either case, the claim remains true. For boundary boxes, the number of boundary edges can exceed the number of interior edges, but only when all (zero or one) interior edges have been taken. Thus, at any time, no box except possibly the one in which Player 2 just played could be completed by an interior edge; all other boxes can be completed only by boundary edges, each of which is incident to only one box. Therefore at no time can any edge complete two boxes simultaneously.

Next we prove that Player 1 completes no boxes during Phase II. By definition, Player 1 will take a box during Phase II only if every box is either completed or one edge from being completed. At such a time, the taken edges must include a spanning tree of the grid, which consists of $2n + 1$ edges (see Figure 3), plus exactly one edge for each completed box (because the cycle formed by each completed box can be broken by a single edge removal). Because we proved that each edge completed at most one box, the number of complete turns must be the number of taken edges minus the number of completed boxes. Thus the number of complete turns must be $2n + 1$, meaning that it is Player 2's turn. Therefore Player 1 completes no boxes during Phase II.

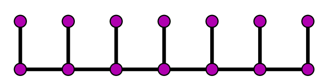

Figure 3: A spanning tree of a $1 \times n$ grid has $2n + 1$ edges.

We claim that Player 1 never completes an end box. An end box has at most one interior edge, so there is only one possible move by Player 1 that could complete the box in Phase I. But when Player 2 plays the top or bottom edge of the box, Player 1 will take the interior edge, before Player 2 could have played the opposite (bottom or top) edge of the box. Therefore this move by Player 1 did not complete the box.

If Player 1 plays first in a non-end box of the board, then we claim that Player 1 will not complete this box; refer to Figure 4. If Player 1 also plays second in this box, then the claim is obvious: Player 1 will play in the box only if it does not complete the box. If Player 2 plays second in the box, then by definition Player 1 will immediately take the remaining interior edge of the box. As this is only the third move in the box, this move does not complete the box. Player 1 will not play the final boundary edge of the box because that would complete the box.

(a) Player 1 starts. (b) Player 2 starts.

Figure 4: Behavior of a non-end box under Player 1's strategy, depending on who takes the first box edge. Non-immediate responses are denoted by $\cdots$.

Finally we show that Player 1 completes at most $\lfloor (n-1)/3 \rfloor$ boxes. As argued above, for Player 1 to complete a box, it must not be an end box and Player 2 must play in it first. Indeed, Player 2 must play in that box again, taking the other boundary edge, or else we would have already entered Phase II. Thus, every box taken by Player 1 can be charged to two moves by Player 2, as well as the two following interior edges taken by Player 1. Furthermore, the completed box means that Player 1 also takes another interior edge (if there is one). Thus every box completed by Player 1 corresponds to an increase in the number of taken interior by at least 3. Therefore Player 1 completes at most $(n-1)/3$ boxes. □

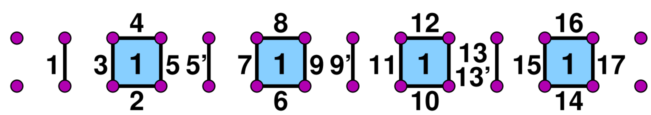

Figure 5: A strategy for Player 2 that causes Player 1's strategy to complete $\lfloor (n-1)/3 \rfloor$ boxes. Edge labels denote turn number.

Figure 5 shows an example where the strategy of Theorem 1 causes Player 1 to take $\lfloor (n-1)/3 \rfloor$ boxes. We conjecture that this strategy is optimal, at least up to additive constants.

Open Problem 1 *Can Player 2 force Player 1 to complete* $\lfloor (n-1)/3 \rfloor$ *boxes?*

3 Misère $2 \times n$

For misère $2 \times n$ Dots-and-Boxes, which player is the initial leader changes with n. In the absence of parity-switching moves, the first player should win for odd n and the second player should win for even n. By Theorem 1, we already know this to be the case for $n = 1$. We have also verified this claim by exhaustive computational search for $n = 2$ and $n = 3$.

A natural strategy for the leading player, generalizing the $1 \times n$ strategy, is the following. If you can complete a box, then take it. (This rule prevents the formation of larger parity-changing cycles.) Otherwise, if there is an untaken internal edge incident to the edge just taken by the other player, then take it. Otherwise, if there is an untaken internal edge, take it. Otherwise, if there is an edge that does not complete a box, take it. Otherwise, take any edge. (The last rule should not arise if the parity remains unchanged.) We have verified that this strategy works for 2×2 but not for 2×3 or larger boards.

In fact, for sufficiently large $2 \times n$ boards, it seems that the nonleading player can force the leading player to take around $3/4$ of the boxes. If this is the case, then either the nonleading player wins, or the leading player must change the parity. We wonder whether such a change in parity (perhaps just once or twice?) can let the leading player guarantee a win.

1×2	—	second player wins by 2 points
1×3	—	second player wins by 3 points
1×4	—	first player wins by 4 points
1×5	—	first player wins by 5 points
1×6	—	first player wins by 6 points
1×7	—	first player wins by 5 points
1×8	—	first player wins by 4 points
1×9	—	first player wins by 3 points
1×10	—	first player wins by 2 points
1×11	—	first player wins by 1 point
1×12	—	draw
1×13	—	second player wins by 1 point
1×14	—	second player wins by 2 points
1×15	—	second player wins by 1 point
1×16	—	draw
1×17	—	first player wins by 1 point
1×18	—	first player wins by 2 points
1×19	—	first player wins by 1 point
1×20	—	first player wins by 2 points
1×21	—	first player wins by 1 point
1×22	—	draw
1×23	—	second player wins by 1 point

1×24	—	draw
1×25	—	second player wins by 1 point
1×26	—	draw
1×27	—	first player wins by 1 point
1×28	—	draw
1×29	—	first player wins by 1 point
1×30	—	draw
1×31	—	first player wins by 1 point
1×32	—	draw
1×33	—	second player wins by 1 point
1×34	—	draw
1×35	—	second player wins by 1 point
1×36	—	draw
1×37	—	first player wins by 1 point
1×38	—	draw
1×39	—	first player wins by 1 point
1×40	—	draw
1×41	—	first player wins by 1 point
1×42	—	draw
1×43	—	second player wins by 1 point
1×44	—	draw
1×45	—	???

Table 1: Who wins in Swedish $1 \times n$ misère Dots-and-Boxes under optimal play, as computed by an exhaustive search.

4 Misère Swedish $1 \times n$

In *Swedish Dots-and-Boxes* [Wil], all boundary edges are initially drawn. In this case, $1 \times n$ misère Dots-and-Boxes has a much more complicated behavior; see Table 1. These results are based on exhaustive computational search.

This game seems particularly interesting because it is very simple, yet is all about the parity switching of who leads. Conceivably, $1 \times n$ Swedish games could also arise in the middle of a $2 \times n$ game, though it is unclear whether this happens under optimal play.

Conjecture 1 *The outcome of misère $1 \times n$ Swedish Dots-and-Boxes is given by Table 1, with periodic behavior starting from $n = 22$ and a period of* 10*.*

This conjecture is supported by different observations. First, it matches the behavior exposed by our exhaustive search, as shown in Table 1. Second, we have a detailed proof that the outcome is correct for a restricted form of the game, in which every edge drawn must form a rectangle of size 1×1, 1×2, 1×3, or 1×4, and the last case only when there is not already another 1×4 rectangle on the board. We believe that the outcome of this restricted game is equivalent to the original one:

Conjecture 2 *For misère $1 \times n$ Swedish Dots-and-Boxes, there exists an optimal play in which both players move so as to form rectangles of size 1×1, 1×2, 1×3, or 1×4 with every edge drawn, with the last case arising only when there is not already a 1×4 rectangle on the board.*

This conjecture is also supported by our exhaustive search: in the games we so analyze, there is always an optimal move of the restricted form, as shown in Table 2. If Conjecture 2 is true, our proof implies Conjecture 1. The outcome under optimal play of the restricted game (and thus also of the unrestricted game if Conjecture 2 holds) is the following eventually periodic sequence:

$$-2, -3, 4, 5, 6, 5, 4, 3, 2, 1, 0, -1, -2, -1, 0, 1, 2, 1, 2, 1, [0, -1, 0, -1, 0, 1, 0, 1, 0, 1]^*.$$

These numbers indicate by how many points the first player wins under optimal play, with a positive number meaning a first-player win, zero meaning a draw, and a negative number meaning a second-player win.

Acknowledgments

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1×2	any
1×3	any
1×4	cut 2
1×5	cut 2
1×6	cut 3
1×7	take all but 6 then cut 3
1×8	take all but 6 then cut 3
1×9	take all but 6 then cut 3
1×10	take all but 6 then cut 3
1×11	take all but 6 then cut 3
1×12	take all but 6 then cut 3
1×13	take all but 6 then cut 3
1×14	cut 2; or take all but 6 then cut 3
1×15	cut 2
1×16	cut 2
1×17	cut 2
1×18	cut 2
1×19	cut 2, 3, . . . , or 9; or take 1 then cut 2
1×20	cut 3, 5, 6, . . . , or 10
1×21	cut 3, 5, 6, . . . , or 10; or take 1 then cut 3, 5, 6, . . . , or 10
1×22	cut 3, 5, 6, . . . , or 11; take 1 then cut 3, 5, 6, . . . , or 10; or take 2 then cut 3, 5, 6, . . . , or 10
1×23	cut 2, 3, . . . , or 11; take 1 then cut 3, 5, 6, . . . , or 11; take 2 then cut 3, 5, 6, . . . , or 10; or take 3 then cut 3, 5, 6, . . . , or 10
1×24	cut 4
1×25	cut 2, 3, . . . , or 12; or take 1 then cut 4
1×26	cut 2 or 4
1×27	cut 2
1×28	cut 2, 3, . . . , or 14; or take 1 then cut 2
1×29	cut 2, 3, . . . , or 14

1×30	cut 2, 3, . . . , or 15; or take 1 then cut 2, 3, . . . , or 14
1×31	cut 3, 5, 6, . . . , or 15
1×32	cut 2, 3, . . . , or 16; or take 1 then cut 3, 5, 6, . . . , or 15
1×33	cut 2, 3, . . . , or 16; take 1 then cut 2, 3, . . . , or 16; or take 2 then cut 3, 5, 6, . . . , or 15
1×34	cut 2, 3, . . . , or 17
1×35	cut 2, 3, . . . , or 17; or take 1 then cut 2, 3, . . . , or 17
1×36	cut 2, 3, . . . , or 18
1×37	cut 2 or 4
1×38	cut 2, 3, . . . , or 19; or take 1 then cut 2 or 4
1×39	cut 2, 3, . . . , or 19
1×40	cut 2, 3, . . . , or 20; or take 1 then cut 2, 3, . . . , or 19
1×41	cut 3, 5, 7, 8, . . . , or 20
1×42	cut 2, 3, . . . , or 21; or take 1 then cut 3, 5, 7, 8, . . . , or 20
1×43	cut 2, 3, . . . , or 21; take 1 then cut 2, 3, . . . , or 21; or take 2 then cut 3, 5, 7, 8, . . . , or 20
1×44	cut 2, 3, . . . , or 22
1×45	cut 2, 3, . . . , or 22; or take 1 then cut 2, 3, . . . , or 22
1×46	cut 2, 3, . . . , or 23
1×47	cut 2 or 4
1×48	cut 2, 3, . . . , or 24; or take 1 then cut 2 or 4
1×49	cut 2, 3, . . . , or 24
1×50	cut 2, 3, . . . , or 25; or take 1 then cut 2, 3, . . . , or 24
1×51	cut 3, 5, 7, 9, 10, . . . , or 25
1×52	cut 2, 3, . . . , or 26; or take 1 then cut 3, 5, 7, 9, 10, . . . , or 25

Table 2: All optimal moves for the first player from the initial configuration of the $1 \times n$ Swedish board, as computed by an exhaustive search. Here "take k" means to complete n boxes, "cut k" means to draw an edge to form a $1 \times k$ rectangle, and an ellipsis (. . .) denotes an interval of consecutive integers. To remove symmetric moves, we omit "cut k" when k is larger than half the current rectangle.