LOGARITHMIC LOWER BOUNDS IN THE CELL-PROBE MODEL*

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Abstract. We develop a new technique for proving cell-probe lower bounds on dynamic data structures. This technique enables us to prove an amortized randomized $\Omega(\lg n)$ lower bound per operation for several data structural problems on n elements, including partial sums, dynamic connectivity among disjoint paths (or a forest or a graph), and several other dynamic graph problems (by simple reductions). Such a lower bound breaks a long-standing barrier of $\Omega(\lg n / \lg\lg n)$ for any dynamic language membership problem. It also establishes the optimality of several existing data structures, such as Sleator and Tarjan's dynamic trees. We also prove the first $\Omega(\log_B n)$ lower bound in the external-memory model without assumptions on the data structure (such as the comparison model). Our lower bounds also give a query-update trade-off curve matched, e.g., by several data structures for dynamic connectivity in graphs. We also prove matching upper and lower bounds for partial sums when parameterized by the word size and the maximum additive change in an update.

Key words. Cell-probe complexity, lower bounds, data structures, dynamic graph problems, partial-sums problem

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1. Introduction. The cell-probe model is perhaps the strongest model of computation for data structures, subsuming in particular the common word-RAM model. We suppose that the memory is divided into fixed-size cells (words), and the cost of an operation is just the number of cells it reads or writes. Typically we think of the cell size as being around $\lg n$ bits long, so that a single cell can address all n elements in the data structure. (Refer to Section 4 for a precise definition of the model.) While unrealistic as a model of computation for actual data structures, the generality of the cell-probe model makes it an important model for lower bounds on data structures.

Previous cell-probe lower bounds for data structures fall into two categories of approaches. The first approach is based on communication complexity. Lower bounds for the predecessor problem [Ajt88, MNSW98, BF02, SV] are perhaps the most successful application of this idea. Unfortunately, this approach can only be applied to problems that are hard even in the static case. It also requires queries to receive a parameter of $\omega(\lg n)$ bits, which is usually interpreted as requiring cells to have $\omega(\lg n)$ bits. For problems that are hard only in the dynamic case, all lower bounds have used some variation of the chronogram method of Fredman and Saks [FS89]. By design, this method cannot prove a trade-off between the query time t_q and the update time t_u better than $t_q \lg t_u = \Omega(\lg n)$, which was achieved for the marked-ancestor problem (and consequently many other problems) in [AHR98]. This limitation on trade-off lower bounds translates into an $\Omega(\lg n / \lg\lg n)$ limitation on lower bounds for both queries and updates provable by this technique. The $\Omega(\lg n / \lg\lg n)$ barrier has been recognized as an important limitation in the study of data structures, and was proposed as a major challenge for future research in a recent survey [Mil99].

This paper introduces a new technique for proving cell-probe lower bounds on dynamic data structures. With this technique we establish an $\Omega(\lg n)$ lower bound for

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either queries or updates in several natural and well-studied problems, in particular, maintaining partial (prefix) sums in an array and dynamic connectivity among disjoint paths (or a forest or a graph). (We detail the exact problems we consider, and all results we obtain, in Section 2; we summarize relevant previous results on these problems in Section 3.) These lower bounds establish the optimality of several data structures, including the folklore $O(\lg n)$ balanced tree data structure for partial sums, and Sleator and Tarjan's dynamic trees data structure (which in particular maintains dynamic connectivity in a forest).

We also prove a trade-off lower bound of $t_q \lg \frac{t_u}{t_q} = \Omega(\lg n)$.[1] This trade-off turns out to be the right answer for our problems, and implies the $\Omega(\lg n)$ bound on the worst of queries and updates. In addition, we can prove a symmetric trade-off $t_u \lg \frac{t_q}{t_u} = \Omega(\lg n)$. As mentioned above, it is fundamentally impossible to achieve such a trade-off using the previous techniques.

We also refine our analysis of the partial-sums problem beyond just the dependence on n. Specifically, we parameterize by n, the number b of bits in a word, and the number δ of bits in an update. Naturally, $\delta \leq b$, but in some applications, δ is much smaller than b. We prove tight upper and lower bounds of $\Theta(\frac{\lg n}{\lg(b/\delta)})$ on the worst of queries and updates. This result requires improvements in both the upper bounds and the lower bounds. In addition, we give a tight query/update trade-off: $t_q \left(\lg \frac{b}{\delta} + \lg \frac{t_u}{t_q}\right) = \Theta(\lg n)$. The tightness of this characterization is particularly unusual given its dependence on five variables.

The main idea behind our lower-bound technique is to organize time (the sequence of operations performed on the data structure) into a complete tree. The heart of the analysis is an encoding/decoding argument that bounds the amount of information transferred between disjoint subtrees of the tree: if few cell are read and written, then little information can be transferred. The nature of the problems of interest requires at least a certain amount of information transfer from updates to queries, providing a lower bound on the number of cells read and written. This main idea is developed first in Section 5 in the context of the partial-sums problem, where we obtain a short (approximately three-page) proof of an $\Omega(\lg n)$ lower bound for partial sums. Compared to the lower bounds based on previous techniques, our technique leads to relatively clean proofs with minimal combinatorial calculation.

We generalize this basic approach in several directions to obtain our further lower bounds. In Section 6, we show how our technique can be extended to handle queries with binary answers (such as dynamic connectivity) instead of word-length answers (such as partial sums). In particular, we obtain an $\Omega(\lg n)$ lower bound for dynamic connectivity in disjoint paths. We also show how to use our lower-bound technique in the presence of nondeterminism or Monte Carlo randomization. In Section 7, we show how our technique can be further extended to handle updates asymptotically smaller than the word size, in particular obtaining lower bounds for the partial-sums problem when $\delta < b$ and for dynamic connectivity in the external-memory model. This last section develops the most complicated form of our technique.

The final few sections contain complementary results to this main flow of the lower-bound technique. In Section 8, we give tight upper bounds for the partial-sums problem. The data structure is based on a few interesting ideas that enable us to eliminate the precomputed tables from previous approaches. In Section 9, we prove

[1]Throughout this paper, $\lg x$ denotes $\log_2(2 + x)$, which is positive for all $x \geq 0$ (an important property in this bound and several others).

some easy reductions from dynamic connectivity to other dynamic graph problems, transferring our lower bounds to these problems. Finally, we conclude in Section 10 with a list of open problems.

2. Results. In this section we give precise descriptions of the problems we consider, our results, and a brief synopsis of how these results compare to previous work. (Section 3 gives a more detailed historical account.)

2.1. The Partial-Sums Problem. This problem asks to maintain an array $A[1..n]$ of n integers subject to the following operations:

UPDATE(k, Δ)**:** modify $A[k] \leftarrow \Delta$.
SUM(k)**:** returns the partial sum $\sum_{i=1}^{k} A[i]$.
SELECT(σ)**:** returns an index i satisfying $\text{sum}(i-1) < \sigma \leq \text{sum}(i)$. To guarantee uniqueness of the answer, we require that $A[i] > 0$ for all i.

Besides n, the problem has several interesting parameters. One parameter is b, the number of bits in a cell (word). We assume that every array element and sum fits in a cell. Also, we assume that $b = \Omega(\lg n)$. Another parameter is δ, the number of bits needed to represent an argument Δ to UPDATE. Naturally, δ is bounded above by b; however, it is traditional (see, e.g. [RRR01]) to consider a separate parameter δ because it is smaller in many applications. We write t_u for the running time of UPDATE, t_q for the running time of SUM, and t_s for the running time of SELECT.

We first study the unrestricted case when $\delta = \Omega(b)$:

THEOREM 2.1. *Consider any cell-probe data structure for the partial-sums problem that may use amortization and Las Vegas randomization. If $\delta = \Omega(b)$, then the following trade-offs hold:*

$$t_q \lg(t_u/t_q) = \Omega(\lg n); \qquad t_u \lg(t_q/t_u) = \Omega(\lg n);$$
$$t_s \lg(t_u/t_s) = \Omega(\lg n); \qquad t_u \lg(t_s/t_u) = \Omega(\lg n).$$

The trade-off curves are identical for the SELECT and SUM operations. The first branch of each trade-off is relevant when queries are faster than updates, while the second branch is relevant when updates are faster. The trade-offs imply the long-sought logarithmic bound for the partial-sums problem: $\max\{t_u, t_q\} = \Omega(\lg n)$. The best previous bound, by Fredman and Saks [FS89], was $t_q \lg(bt_u) = \Omega(\lg n)$, implying $\max\{t_u, t_q\} = \Omega(\lg n / \lg b)$. The trade-off curves between t_u and t_q also hold in the group model of computation, where elements of the array come from a black-box group and time is measured as the number of algebraic operations. The best previous bound for this model was $\Omega(\lg n / \lg \lg n)$, also by Fredman and Saks [FS89].

A classic result achieves $t_u = t_q = t_s = O(\lg n)$. For the SUM query, our entire trade-off curve can be matched (again, this is folklore; see the next section on previous work). For SELECT, the trade-offs cannot be tight for the entire range of parameters, because even for polynomial update times, there is a superconstant lower bound on the query time for the predecessor problem [Ajt88, SV].

We also analyze the case $\delta = o(b)$. We first give the following lower bounds:

THEOREM 2.2. *Consider any cell-probe data structure for the partial-sums problem using amortization and Las Vegas randomization. The following trade-offs hold:* $t_q \left(\lg \frac{b}{\delta} + \lg \frac{t_u}{t_q}\right) = \Omega(\lg n)$ *and* $t_s \left(\lg \frac{b}{\delta} + \lg \frac{t_u}{t_s}\right) = \Omega(\lg n)$

In this case, we cannot prove a reverse trade-off (when updates are faster than queries). This trade-off implies the rather interesting lower bound $\max\{t_u, t_q\} =$

$\Omega(\frac{\lg n}{\lg(b/\delta)})$, and similarly for t_s. When $\delta = \Theta(b)$, this gives the same $\Omega(\lg n)$ as before. We give new matching upper bounds:

THEOREM 2.3. *There exists a data structure for the partial-sums problem achieving* $t_u = t_q = t_s = O\left(\frac{\lg n}{\lg(b/\delta)}\right)$. *The data structure runs on a Random Access Machine, is deterministic, and achieves worst-case bounds.*

Our upper bounds can handle a slightly harder version of the problem, where UPDATE(i, Δ) has the effect $A[i] \leftarrow A[i] + \Delta$. Thus, we are not restricting each $A[i]$ to δ bits, but just mandate that they don't grow by more than a δ-bit term at a time. Several previous results [Die89, RRR01, HSS03] achieved $O(\lg n/\lg\lg n)$ bounds for $\delta = O(\lg\lg n)$. None of these solutions scale well with δ or b, because they require large precomputed tables.

This result matches not only the previous lower bound on the hardest operation, but actually helps match the entire trade-off of Theorem 2.2. Indeed, the trade-off lower bound shows that there is effectively no interesting trade-off when $\delta = o(b)$: when $\frac{b}{\delta} \geq \frac{t_u}{t_q}$, the tight bound is $t_q = \Theta(\frac{\lg n}{\lg(b/\delta)})$, matched by our structure; when $\frac{b}{\delta} < \frac{t_u}{t_q}$, the tight bound is $t_q = \Theta(\frac{\lg n}{\lg(t_u/t_q)})$, matched by the classic result which does not depend on δ. Thus, we obtain an unusually precise understanding of the problem, having a trade-off that is tight in all five parameters.

2.2. Dynamic Connectivity. This problem asks to maintain an undirected graph with a fixed set of n vertices subject to the following operations:

INSERT(u, v)**:** insert an edge (u, v) into the graph.
DELETE(u, v)**:** delete the edge (u, v) from the graph.
CONNECTED(u, v)**:** test whether u and v lie in the same connected component.

We write t_q for the running time of CONNECTED, and t_u for the running time of INSERT and DELETE. It makes the most sense to study this problem in the cell-probe model with $O(\lg n)$ bits per cell, because every quantity in the problem occupies $O(\lg n)$ bits.

We prove the following lower bound:

THEOREM 2.4. *Any cell-probe data structure for dynamic connectivity satisfies the following trade-offs:* $t_q \lg(t_u/t_q) = \Omega(\lg n)$ *and* $t_u \lg(t_q/t_u) = \Omega(\lg n)$. *These bounds hold under amortization, nondeterministic queries, and Las Vegas randomization, or under Monte Carlo randomization with error probability* $n^{-\Omega(1)}$. *These bounds hold even if the graph is always a disjoint union of paths.*

This lower bound holds under very broad assumptions. It allows for nondeterministic computation or Monte Carlo randomization (with polynomially small error), and holds even for paths (and thus for trees, plane graphs etc.). The trade-offs we obtain are identical to the partial-sums problem. In particular, we obtain that $\max\{t_u, t_q\} = \Omega(\lg n)$.

An upper bound of $O(\lg n)$ for trees is given by the famous dynamic trees data structure of Sleator and Tarjan [ST83]. In addition, the entire trade-off curve for $t_u = \Omega(t_q)$ can be matched for trees. For general graphs, Thorup [Tho00] gave an almost-matching upper bound of $O(\lg n(\lg\lg n)^3)$. For any $t_u = \Omega(\lg n(\lg\lg n)^3)$, his data structure can match our trade-off.

Dynamic connectivity is perhaps the most fundamental dynamic graph problem. It is relatively easy to show by reductions that our bounds hold for several other dynamic graph problems. Section 9 describes such reductions for deciding connectivity of the entire graph, minimum spanning forest, and planarity testing. Many data structural problems on undirected graphs have polylogarithmic solutions, so our bound is arguably interesting for these problems. Some problems have logarithmic solutions for special cases (such as plane graphs), and our results prove optimality of those data structures.

We also consider dynamic connectivity in the external-memory model. Let B be the page (block) size, i.e., the number of $(\lg n)$-bit cells that fit in one page. We prove the following lower bound:

THEOREM 2.5. *A data structure for dynamic connectivity in the external-memory model with page size B must satisfy $t_q \left(\lg B + \lg \frac{t_u}{t_q}\right) = \Omega(\lg n)$. This bound allows for amortization, Las Vegas randomization, and nondeterminism, and holds even if the graph is always a disjoint union of paths.*

Thus we obtain a bound of $\max\{t_u, t_q\} = \Omega(\log_B n)$. Although bounds of this magnitude are ubiquitous in the external-memory model, our lower bound is the first that holds in a general model of computation, i.e., allowing data items to be manipulated arbitrarily and just counting the number of page transfers. Previous lower bounds have assumed the comparison model or indivisibility of data items.

It is possible to achieve an $O(\log_B n)$ upper bound for a forest, by combining Euler tour trees with buffer trees [Arg03]. As with the partial-sums problem, this result implies that our entire trade-off is tight for trees: for $B \geq t_u/t_q$, this solution is optimal; if the term in t_u/t_q dominates, we use the classic trade-off, which foregoes the benefit of memory pages.

3. Previous Work. In this section we detail the relevant history of cell-probe lower bounds in general and the specific problems we consider.

3.1. Cell-Probe Lower Bounds. Fredman and Saks [FS89] were the first to prove cell-probe lower bounds for dynamic data structures. They developed the chronogram technique, and used it to prove a lower bound of $\Omega(\lg n/\lg b)$ for the partial-sums problem in $\mathbb{Z}/2\mathbb{Z}$ (integers modulo 2, where elements are bits and addition is equivalent to binary exclusive-or). This bound assumes $b \geq \lg n$ so that an index into the n-element array fits in a word; for the typical case of $b = \Theta(\lg n)$, it implies an $\Omega(\lg n/\lg\lg n)$ lower bound. Fredman and Saks also obtain a trade-off of $t_q = \Omega(\frac{\lg n}{\lg b + \lg t_u})$.

There has been considerable exploration of what the chronogram technique can offer. Ben-Amram and Galil [BAG01] reprove the lower bounds of Fredman and Saks in a more formalized framework, centered around the concepts of problem and output variability. Using these ideas, they show in [BAG02] that the lower bound holds even if cells have infinite precision, but the set of operations is restricted.

Miltersen et al. [MSVT94] observe that there is a trivial reduction from the partial-sums problem in $\mathbb{Z}/2\mathbb{Z}$ to dynamic connectivity, implying an $\Omega(\lg n/\lg\lg n)$ lower bound for the latter problem. Independently, Fredman and Henzinger [FH98] observe the same reduction, as well as some more complex reductions applying to connectivity in plane graphs and dynamic planarity testing. Husfeldt and Rauhe [HR03] show slightly stronger results using the chronogram technique. They prove that the lower bound holds even for nondeterministic algorithms, and even in a promise version of the problem in which the algorithm is told the requested sum to a ± 1 precision. These improved results make it possible to prove reductions to various other problems [HR03, HRS96].

Alstrup, Husfeldt, and Rauhe [AHR98] give the only previous improvement to the bounds of Fredman and Saks, by proving a stronger trade-off of $t_q \lg t_u = \Omega(\lg n)$. This bound is the best trade-off provable by the chronogram technique. However, it still cannot improve beyond $\max\{t_u, t_q\} = \Omega(\lg n / \lg\lg n)$. The problem they considered was the partial-sums problem generalized to trees, where a query asks for the sum of a root-to-leaf path. (This is a variation of the more commonly known marked-ancestor problem.) Their bound is tight for balanced trees; for arbitrary trees, our lower bound shows that $\Theta(\lg n)$ is the best possible.

Miltersen [Mil99] surveys the field of cell-probe complexity, and advocates "dynamic language membership" problems as a standardized framework for comparing lower bounds. Given a language L that is polynomial-time decidable, the *dynamic language membership problem* for L is defined as follows. For any given n (the problem size), maintain a string $w \in \{0,1\}^n$ under two operations: flip the i^{th} bit of w, and report whether $w \in L$. Through its minimalism, this framework avoids several pitfalls in comparing lower bounds. For instance, it is possible to prove very high lower bounds in terms of the number of cells in the problem representation (which, misleadingly, is often denoted n), if the cells are large [Mil99]. However, these lower bounds are not very interesting because they assume exponential-size cells. In terms of the number of bits in the problem representation, all known lower bounds do not exceed $\Omega(\lg n / \lg\lg n)$.

Miltersen proposes several challenges for future research, two of which we solve in this paper. One such challenge was to prove an $\Omega(\lg n)$ lower bound for the partial-sums problem. Another such challenge, listed as one of three "big challenges", was to prove a lower bound of $\omega(\lg n / \lg\lg n)$ for a dynamic language membership problem. We solve this problem because dynamic connectivity can be phrased as a dynamic language membership problem [Mil99].

3.2. The Partial-Sums Problem in Other Models. The partial-sums problem has been studied since the dawn of data structures, and has served as the prototypical problem for the study of lower bounds. Initial efforts concentrated on algebraic models of computation. In the semigroup or group models, the elements of the array come from a black-box (semi)group. The algorithm can only manipulate the Δ inputs through additions and, in the group model, subtractions; all other computations in terms of the indices touched by the operations are free.

In the semigroup model, Fredman [Fre81] gives a tight logarithmic bound. However, this bound is generally considered weak, because updates have the form $A[i] \leftarrow \Delta$. Because additive inverses do not exist, such an update invalidates all memory cells storing sums containing the old value of $A[i]$. When updates have the form $A[i] \leftarrow A[i] + \Delta$, Yao [Yao85] proved a lower bound of $\Omega(\lg n / \lg\lg n)$. Finally, Hampapuram and Fredman [HF98] proved an $\Omega(\lg n)$ lower bound for this version of the problem; their bound holds even for the offline problem. In higher dimensions, Chazelle [Cha97] gives a lower bound of $\Omega((\lg n / \lg\lg n)^d)$, which also holds even for the offline problem.

In the group model, the best previous lower bound of $\Omega(\lg n / \lg\lg n)$ is by Fredman and Saks [FS89]. A tight logarithmic bound (including the lead constant) was given by [Fre82] for the restricted class of "oblivious" algorithms, whose behavior can be described by matrix multiplication. For the offline problem, Chazelle [Cha97] gives a lower bound of $\Omega(\lg\lg n)$ per operation; this is exponentially weaker than the best known upper bound. No better lower bounds are known in higher dimensions.

3.3. Upper Bounds for the Partial-Sums Problem. An easy $O(\lg n)$ upper bound for partial sums is to maintain a balanced binary tree with the elements of A in the leaves, augmented to store partial sums for each subtree. A simple variation of this scheme yields an implicit data structure occupying exactly

n memory locations [Fen94]. For the SUM query, it is easy to obtain good trade-offs. Using trees with branching factor B, one can obtain $t_q = O(\log_B n)$ and $t_u = O(B \log_B n)$, or $t_q = O(B \log_B n)$ and $t_u = O(\log_B n)$. These bounds can be rewritten as $t_q \lg \frac{t_u}{t_q} = O(\lg n)$, or $t_u \lg \frac{t_q}{t_u} = O(\lg n)$, respectively, which matches our lower bound for the case $\delta = \Theta(b)$, and for the group model. For SELECT queries, one cannot expect to achieve the same trade-offs, because even for a polynomial update time, there is a superconstant lower bound on the predecessor problem [BF02]. Exactly what trade-offs are possible remains an open problem.

Dietz [Die89] considers the partial-sums problem with SUM queries on a RAM, when $\delta = o(b)$. He achieves $O(\lg n/\lg\lg n)$ running times provided that $\delta = O(\lg\lg n)$. Raman, Raman, and Rao [RRR01] show how to support SELECT in $O(\lg n/\lg\lg n)$, again if $\delta = O(\lg\lg n)$. For $t_u = \Omega(\lg n/\lg\lg n)$, the same δ, and SUM queries, they give a trade-off of $t_q = O(\log_{t_u} n)$. They achieve the same trade-off for SELECT queries, when $\delta = 1$. Hon, Sadakane, and Sung [HSS03] generalize the trade-off for SELECT when $\delta = O(\lg\lg n)$. All of these results do not scale well with b or δ because of their use of precomputed tables.

3.4. Upper Bounds for Dynamic Connectivity. For forests, Sleator and Tarjan's classic data structure for dynamic trees [ST83] achieves an $O(\lg n)$ upper bound for dynamic connectivity. A simpler solution is given by Euler tour trees [HK99]. This data structure can achieve a running time of $t_q = O(\frac{\lg n}{\lg(t_u/t_q)})$, matching our lower bound.

For general graphs, the first to achieve polylogarithmic time per operation were Henzinger and King [HK99]. They achieve $O(\lg^3 n)$ per update, and $O(\lg n/\lg\lg n)$ per query, using randomization and amortization. Henzinger and Thorup [HT97] improve the update bound to $O(\lg^2 n)$. Holm, de Lichtenberg, and Thorup [HdLT01] give a simple deterministic solution with the same amortized running time: $O(\lg^2 n)$ per update and $O(\lg n/\lg\lg n)$ per query. The best known result in terms of updates is by Thorup [Tho00], achieving nearly logarithmic running times: $O(\lg n(\lg\lg n)^3)$ per update and $O(\lg n/\lg\lg\lg n)$ per query. This solution is only a factor of $(\lg\lg n)^3$ away from our lower bound. Interestingly, all of these solutions are on our trade-off curve. In fact, for any $t_u = \Omega(\lg n(\lg\lg n)^3)$, Thorup's solution can achieve $t_q = O(\frac{\lg n}{\lg(t_u/q_t)})$, showing that our trade-off curve is optimal for this range of t_u.

For plane graphs, Eppstein et al. [EIT+92] give a logarithmic upper bound. Plane graphs are planar graphs with a given topological planar embedding, specified by the order of the edges around each vertex. Our lower bound holds for such graphs, proving the optimality of this data structure.

4. Models. The cell-probe model is a nonuniform model of computation. The memory is represented by a collection of cells. Operations are handled by an algorithm which can read and write cells from the memory; all computation is free, and the internal state is unbounded. However, the state is lost at the end of an operation. Because state is not bounded, it can be assumed that all writes happen at the end of the operation. If cells have b bits, we restrict the number of cells to 2^b, ensuring that a pointer can be represented in one cell. This restriction is a version of the standard *transdichotomous assumption* frequently made in the context of the word RAM, and is therefore natural in the cell-probe model as well.

We extend the model to allow for nondeterministic computation, in the spirit of [HR03]. Boolean queries can spawn any number of independent execution threads; the overall result is an accept ("yes" answer) precisely if at least one thread accepts.

The running time of the operation is the running time of the longest thread. Rejecting threads may not write any cells; accepting threads may, as long as all accepting threads write exactly the same values. Because of this restriction, and because updates are deterministic, the state of the data structure is always well-defined.

All lower bounds in this paper hold under Las Vegas randomization, i.e., zero-error randomization. We consider a model of randomization that is particularly easy to reason about in the case of data structures. When the data structure is created, a fixed subset of, say, 2^{b-1} cells is initialized to uniformly random values; from that point on, everything is deterministic. This model can easily simulate other models of randomization, as long as the total running time is at most 2^{b-1} (which is always the case in our lower bounds); the idea is that the data structure maintains a pointer to the next random cell, and increments the pointer upon use. For nondeterministic computation, all accepting threads increment the pointer by the largest number of coins that could be used by a thread (bounded by the running time). Using this model, one can immediately apply the easy direction of Yao's minimax principle [Yao77]. Thus, for any given distribution of the inputs, there is a setting of the random coins such that the amortized running time, in expectation over the inputs, is the same as the original algorithm, in expectation over the random coins. Using the nonuniformity in the model, the fixed setting of the coins can be hardwired into the algorithm.

We also consider Monte Carlo randomization, i.e., randomization with two-sided error. Random coins are obtained in the same way, but now the data structure is allowed to make mistakes. We do not allow the data structure to be nondeterministic. In this paper, we are concerned only with error probabilities of $n^{-\Omega(1)}$; that is, the data structure should be correct with high probability. Note that by holding a constant number of copies of the data structure and using independent coins, the exponent of n can be increased to any desired constant. In the data-structures world, it is natural to require that data structures be correct with high probability, as opposed to the bounded-error restriction that is usually considered in complexity theory. This is because we want to guarantee correctness over a large sequence of operations. In addition, boosting the error from constant to n^{-c} requires $O(\lg n)$ repetitions, which is usually not significant for an algorithm, but is a significant factor in the running time of a data-structure operation.

5. Lower Bounds, Take One. In this section, we give the intuition behind our approach and detail a simple form of it that allows us to prove an $\Omega(\lg n)$ lower bound on the partial-sums problem when $\delta = \Theta(b)$, which is tight in this case. This proof serves as a warmup for the more complicated results in Sections 6 and 7.

5.1. General Framework. We begin with the framework for our lower bounds in general terms. Consider a sequence of data-structure operations $A_1, A_2, \ldots, A_m$, where each A_i incorporates all information characterizing operation i, i.e., the operation type and any parameters for that type of operation. Upon receiving request A_i, the data structure must produce an appropriate response. The information gathered by the algorithm during a query (by probing certain cells) must uniquely identify the correct answer to the query, and thus must encode sufficient information to do so.

To establish the lower bounds of this paper, we establish lower bounds for a simpler type of problem. Consider two adjacent intervals of operations: $A_i, \ldots, A_{j-1}$ and $A_j, \ldots, A_k$. At all times, conceptually associate with each memory cell a *chronogram* [FS89], i.e., the index t of the operation A_t during which the memory cell was last modified. Now consider all read instructions executed by the data structure during operations $A_j, \ldots, A_k$ that access cells with a chronogram in the interval $[i, j-1]$.

In other words, we consider the set of cells written during the time interval $[i, j-1]$ and read during the interval $[j, k]$ before they are overwritten. All *information transfer* from time interval $[i, j-1]$ to time interval $[j, k]$ must be encoded in such cells, and must be executed by such cell writes and reads. If the queries from the interval $[j, k]$ depend on updates from the interval $[i, j-1]$, all the information characterizing this dependency must come from these cell probes, because an update happening during $[i, j-1]$ cannot be reflected in a cell written before time i. The main technical part of our proofs is to establish a lower bound on the amount of information that must be transferred between two time intervals, which implies a corresponding lower bound on the number of cells that must be written and read to execute such transfer. Such bounds will stem from an encoding argument, in conjunction with a simple information-theoretic analysis.

Next we show how to use such a lower bound on the information transfer between two adjacent intervals of operations to prove a lower bound on the data structural problems we consider. Consider a binary tree whose leaves represent the entire sequence of operations in time order. Each node in the tree has an associated time interval of operations, corresponding to the subtree rooted at that node. We can obtain two adjacent intervals of operations by, for example, considering the two nodes with a common parent. For every node in the tree, we define the *information transfer through that node* to be the number of read instructions executed in the subtree of the node's right child that read data written by (i.e., cells last written by) operations in the subtree of the node's left child. The lower bound described above provides a lower bound on this information transfer, for every node. We combine these bounds into a lower bound on the number of cell probes performed during the entire execution by simply summing over all nodes.

To show that this sum of individual lower bounds is indeed an overall lower bound, we make two important points. First, we claim that we are not double counting any read instructions. Any read instruction is characterized by the time when it occurs and the time when the location was last written. Such a read instruction is counted by only one node, namely, the lowest common ancestor of the read and write times, because the write must happen in the left subtree of the node, and the read must happen in the right subtree. The second point concerns the correctness of summing up individual lower bounds. This approach works for the arguments in this paper, because all lower bounds hold in the average case under the same probability distribution for the operations. Therefore, we can use linearity of expectation to break up the total number of read instructions performed on average into these distinct components. Needless to say, worst-case lower bounds could not be summed in this way.

The fact that our lower bounds hold in the average case of an input distribution has another advantage: the same lower bound holds in the presence of Las Vegas randomization. The proofs naturally allow the running time to be a random variable, depending on the input. By the easy direction of the minimax principle, a Las Vegas randomized data structure can be converted into a deterministic data structure that on a given random distribution of the inputs achieves the same expected running time.

This line of argument has an important generalization that we use for proving trade-off lower bounds. Instead of considering a binary tree, we can consider a tree of arbitrary degree. Then we may consider the information transfer either between any node and all its left siblings, or between any node and all its right siblings. Neither of these strategies double counts read instructions, because a read instruction is counted only for a node immediately below the lowest common ancestor of the read and write times.

5.2. An Initial Bound for the Partial-Sums Problem. We are now ready to describe an initial lower bound for the partial-sums problem, which gives a clear and concise materialization of the general approach from the previous section. We will prove a lower bound of $\Omega(\frac{\delta}{b}\lg n)$, which is tight (logarithmic) for the special case of $\delta = \Theta(b)$. The bound from this section only considers SUM queries, and does not allow nondeterminism.

It will be useful to analyze the partial-sums problem over an arbitrary group with at least 2^δ elements. Our proof will not use any knowledge about the group, except the quantity δ. Naturally, the data structure is allowed to know the group; in fact, the data structure need only work for one arbitrary choice of group. In particular, the lower bound will hold for the group $\mathbb{Z}/2^\delta\mathbb{Z}$, the group of δ-bit integers with addition modulo 2^δ. A solution to the original partial-sums problem also gives a solution to the problem over this group, as long as we can avoid overflowing a cell in the original problem. To guarantee this, it suffices that $\delta + \lg n < b$. By definition of the model, we always have $\lg n \leq b$ and $\delta \leq b$, so we can avoid overflow by changing only constant factors.

We consider a sequence of $m = \Omega(\sqrt[3]{n})$ operations, where m is a power of two. Operations alternate between updates and queries. We choose the index in the array touched by the operation uniformly at random. If the operation is an update, we also choose the value Δ uniformly at random. This notion of random updates and queries remains unchanged in our subsequent lower bounds, but the pattern of alternating updates and queries changes. Our lemmas do not assume anything about which operations are updates or queries, making it possible to reuse them later.

Our lower bound is based on the following lemma analyzing intervals of operations.

LEMMA 5.1. *Consider two adjacent intervals of operations such that the left interval contains L updates, the right interval contains L queries, and overall the intervals contain $O(\sqrt[3]{n})$ operations. Let c be the number of read instructions executed during the second interval that read cells last written during the first interval. Then $E[c] = \Omega(\frac{\delta}{b}L)$.*

Before we embark on a proof of the lemma, we show how it implies our logarithmic lower bound. As in the framework discussion, we consider a complete binary tree with one leaf per operation. For every node v, we analyze the information transfer through v, i.e., the read instructions executed in the subtree of v's right child that access cells with a chronogram in the subtree of v's left child. If v is on the $\frac{1}{3}\lg n$ bottommost levels, the conditions of the lemma are satisfied, with L being a quarter of the number of leaves under v. Then, the information transfer through v is $\Omega(L\frac{\delta}{b})$ on average. As explained in the framework discussion, we can simply sum these bounds for all nodes to get a lower bound for the execution time. The information transfer through all nodes on a single level is $\Omega(m\frac{\delta}{b})$ in expectation (because these subtrees are disjoint). Over $\frac{1}{3}\lg n$ levels, the lower bound is $\Omega(m\frac{\delta}{b}\lg n)$, or amortized $\Omega(\frac{\delta}{b}\lg n)$ per operation.

5.3. Interleaving Between Two Intervals. The lower bound for two adjacent intervals of operations depends on the interleaving between the indices updated and queried in the two intervals. More precisely, we care about the indices $a_1, a_2, \ldots$ touched by updates during the left interval of time, and the indices $b_1, b_2, \ldots$ queried during the right interval. By relabeling, assume that $a_1 \leq a_2 \leq \cdots$ and $b_1 \leq b_2 \leq \cdots$. We define the *interleaving number* l to be the number of indices i such that, for some index j, $a_i < b_j \leq a_{i+1}$. In words, the interleaving number counts transitions from runs of a's to runs of b's when merging the two sorted lists of indices.

LEMMA 5.2. *Consider two adjacent intervals of operations such that the left interval contains L updates, the right interval contains L queries, and overall the intervals contain $O(\sqrt[3]{n})$ operations. Then the interleaving between the two intervals satisfies $E[l] = \Theta(L)$ and, with probability $1 - o(1)$, no index is touched by more than one operation.*

Proof. By the birthday paradox, the expected number of indices touched more than once is at most $O((\sqrt[3]{n})^2) \cdot \frac{1}{n} = O(n^{-1/3})$. By Markov's inequality, all indices are unique with probability $1 - O(n^{-1/3})$. Because $l \leq L$, it suffices to prove the lower bound. We show $E[l \mid \text{all indices are unique}] = \Omega(L)$. Because the condition is met with $\Omega(1)$ probability, $E[l] = \Omega(L)$. Fix the set S of $2L$ relevant indices arbitrarily. It remains to randomly designate L of these to be updates from the left interval, and the rest of S to be queries from the right interval. Then l is the number of transitions from updates to queries, as we read S in order. The probability that a transition happens on any fixed position is $\frac{1}{4}$, so by linearity of expectation, $E[l \mid S] = \Omega(L)$. Because this bound holds for any S, we can remove the conditioning. □

The following information-theoretic lemma will be used throughout the paper, by comparing the lower bound it gives with upper bounds given by various encoding algorithms. For an introduction to information theory, we refer the reader to [CT91]. Remember that we are considering the partial-sums problem over an arbitrary group with at least 2^δ elements.

LEMMA 5.3. *Consider two adjacent intervals of operations such that the left interval contains L updates, the right interval contains L queries, and overall the intervals contain $O(\sqrt[3]{n})$ operations. Let G be the random variable giving the indices touched by every operation, and giving the Δ values for all updates except those in the left interval. Let S be the random variable giving all partial sums queried in the right interval. Then $H(S \mid G) = \Omega(L\delta)$.*

Proof. Fix $G = g$ to an arbitrary value, such that no index is touched twice in the two intervals. Let l be the interleaving between the two intervals (l is a function of g). Let U denote the set of indices updated in the left interval. By the definition of the interleaving number, there must exist l queries in the right interval to indices $q_1 < q_2 < \cdots < q_l$ such that $U \cap [q_{t-1}+1, q_t] \neq \emptyset$ for each $t \geq 1$, where q_0 is taken to be $-\infty$. Now let us consider the partial sums queried by these l queries, which we denote $S_1, S_2, \ldots, S_l$. The terms of these sums are elements of the array $A[1\,..\,n]$ at the time the query is made. Some elements were set by updates before the first interval, or during the second interval, so they are constants for $G = g$. However, each S_t contains a random term in $[q_{t-1} + 1, q_t]$, which comes from an update from the first interval. This element was not overwritten by a fixed update from the second interval because, by assumption, no index was updated twice. Then each S_t will be a random variable uniformly distributed in the group: even if we condition on arbitrary values for each but one of the random terms, the sum remains uniformly random in the group because of the existence of inverses. Furthermore, the random variables will be independent, because S_t contains at least one random term that was not present in any S_r with $r < t$ (namely, the term in $[q_{t-1} + 1, q_t]$). Then $H((S_1, \ldots, S_l) \mid G = g) = l\delta$. The variable S entails $S_1, \ldots, S_l$, so $H(S \mid G = g) \geq l\delta$. By Lemma 5.2, $E[l] = \Omega(L)$. Furthermore, with probability $1 - o(1)$, a random G leads to no index being updated twice in the two intervals, so the above analysis applies. Then $H(S \mid G) = \Omega(L\delta)$. □

5.4. Proof of Lemma 5.1. We consider two adjacent intervals of time, the first spanning operations $[i, j-1]$ and the second spanning operations $[j, k]$. We propose an encoding for the partial sums queried in $[j, k]$ given the value of G, and compare its size to the $\Omega(L\delta)$ lower bound of Lemma 5.3. Our encoding is simply the list of addresses and contents of the cells probed in the right interval that were written in the left interval. Thus, we are proposing an encoding of expected size $E[c] \cdot 2b$ bits, proving that $E[c] = \Omega(L\frac{\delta}{b})$. It should be noted that c is a random variable, because the algorithm can make different cell probes for different update parameters.

To recover the partial sums from this encoding, we begin by running the algorithm for the time period $[1, i-1]$; this is possible because all operations before time i are known given G. We then skip the time period $[i, j-1]$ and run the algorithm for the time period $[j, k]$, which will return the partial sums queried during this time. To see why this is possible, notice that a read instruction issued during time period $[j, k]$ falls into one of three categories, depending on the time t_w when the cell was written:

$t_w \geq j$: We can recognize this case by maintaining a list of memory locations written during the simulation; the data is immediately available.

$i \leq t_w < j$: The contents of the memory location is available as part out encoding; we can recognize this case by examining the set of addresses in the encoding.

$t_w < i$: This is the default case, if we failed to satisfy the previous conditions. The contents of the cell is determined from the state of the memory upon finishing the first simulation up to time $i-1$.

5.5. Obtaining Trade-Off Lower Bounds. We now show how our framework can be used to derive trade-off lower bounds. In a nutshell, we consider instances where the cheaper operation is performed more frequently, so that the total cost of queries matches the total cost of updates. Then, we analyze the sequence of operations by considering a tree with a higher branching factor.

Assume there exists a data structure with amortized expected running times bounded by t_u for updates and t_q for queries. Our hard instance consists of blocks of $t_u + t_q$ operations. Each block contains t_q updates and t_u queries; the order inside a block is irrelevant. We generate the arguments to updates and queries randomly as before. Let $B = 2 \cdot \max\left\{\frac{t_u}{t_q}, \frac{t_q}{t_u}\right\}$. We prove below that the expected amortized cost of a block is $\Omega\left(\max\{t_u, t_q\}\frac{\delta}{b}\log_B n\right)$. On the other hand, the expected amortized cost of a block is at most $2t_u t_q$. This implies $\frac{t_u t_q}{\max\{t_u, t_q\}} = \Omega\left(\frac{\delta}{b}\log_B n\right)$, so $\min\{t_u, t_q\} \cdot \lg \frac{\max\{t_u,t_q\}}{\min\{t_u,t_q\}} = \Omega\left(\frac{\delta}{b}\lg n\right)$. This is the desired trade-off, which is tight when $\delta = \Theta(b)$.

To prove the lower bound on blocks, consider a balanced B-ary tree in which the leaves correspond to blocks. We let the total number of blocks be $m = \Theta(\sqrt[6]{n})$. If $\max\{t_u, t_q\} = \Omega(\sqrt[6]{n})$, our lower bound states that $\min\{t_u, t_q\} = \Omega(1)$, so there is nothing to prove. Thus, we can assume $t_u + t_q = O(\sqrt[6]{n})$, which bounds the number of operations in a block. Then, the total number of operations is $O(\sqrt[3]{n})$, satisfying one of the conditions of Lemma 5.1.

For the case $t_u \geq t_q$, we are interested in the information transfer between each node and its left siblings. The subtree of the node defines the right interval of operations, and the union of the subtrees of all left siblings defines the left interval. Let L be the number of blocks in the right interval. We make a claim only regarding nodes that are in the right half of their parent's children. In this case, the number of blocks in the left interval is at least $\frac{B}{2}L$. Then, the number of queries in the right interval is Lt_u, while the number of updates in the left interval is at least $L\frac{t_u}{t_q}t_q = Lt_u$. We can then apply Lemma 5.1; having more updates in the left interval cannot decrease the bound, because moving the beginning of the left interval earlier can only increase the number of cell probes that are counted. Therefore, the expected number of cell probes associated with this node is $\Omega(Lt_u \frac{\delta}{b})$. Now we sum the lower bounds for all nodes on a level, and obtain that the number of cell probes associated with that level is $\Omega(mt_u \frac{\delta}{b})$. Summing over all levels, we get an amortized lower bound of $\Omega(t_u \frac{\delta}{b} \log_B n)$ per block, as desired.

For the case $t_u < t_q$, we apply a symmetric argument. We analyze the information transfer between any node, giving the left interval, and all its right siblings, giving the right interval. For nodes in the first half of their parent's children, the left interval contains Lt_q updates, while the right interval contains at least Lt_q queries. By Lemma 5.1, the expected number of cell probes associated with this node is $\Omega(Lt_q \frac{\delta}{b})$. Thus, the number of cell probes associated with a level is $\Omega(mt_q \frac{\delta}{b})$, and the amortized bound per block is $\Omega(t_q \frac{\delta}{b} \log_B n)$.

5.6. Refinements. First note that our lower bounds so far depend only on randomness in the update parameters Δ, and not on randomness in the update or query indices. Indeed, the value of G is irrelevant, except for the interleaving number that it yields. It follows that the logarithmic lower bound and the trade-off lower bound are also true for sequences of operations in which we fix everything except the Δ parameters, as long as such sequences have a high sum of the interleaving numbers of each node. Our application of Lemma 5.2 can be seen as a probabilistic proof that such bad sequences exist.

The prototypical deterministic sequence with high total interleaving is the bit-reversal permutation. For any n that is a power of two, consider the permutation $\pi : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ that takes i to the integer obtained by reversing i's $\log_2 n$ bits. The corresponding access sequence consists of n pairs of UPDATE and SUM, the i^{th} pair touching index $\pi(i)$. The bit-reversal permutation underlies the Fast Fourier Transform algorithm. It also gives an access sequence that takes $\Omega(\lg n)$ amortized time for any binary search tree [Wil89]. Finally, it was used to prove an $\Omega(\lg n)$ bound for the partial-sums problem in the semigroup model [HF98]. To see why this permutation has high total interleaving, consider the following recursive construction. The permutation π' of order $2n$ is obtained from a permutation π of order n by the rules: $\pi'(i) = 2 \cdot \pi(i)$, $\pi'(i+n) = 2 \cdot \pi(i) + 1$, for $i \in \{0, 1, \ldots, n-1\}$. Each level of the recursion adds an interleaving of n between the left and right halves, so the total interleaving is $\Theta(n \lg n)$.

The fact that our lower bound holds for fixed sequences of operations implies the same lower bound in the group model. A solution in the group model handles every UPDATE and SUM by executing a sequence of additions on cells containing abstract elements from the group. The cells touched by these additions depend only on the indices touched by queries and updates, because the data structure treats the group as a black box, and cannot examine the Δ's. So if we know a priori the sequence of indices touched by queries and updates, we can implement the same solution in the cell-probe model for the group $\mathbb{Z}/2^b\mathbb{Z}$; because the Δ's are unrestricted elements of the group, $\delta = b$. The group additions can be hard-wired into our solution for the cell-probe model through nonuniformity, and cell probes are needed only to execute the actual additions.

5.7. Duality of Lower and Upper Bounds. Recall the classic upper bound for the partial-sums problem. We maintain a tree storing the elements of the array in order in the leaves. Each node stores the sum of all leaves in its subtree. An update adds Δ to the subtree sums along the root-to-leaf path of the touched element. A query traverses the root-to-leaf path of the element and reports the sum of all subtrees to the left of the path.

Our lower bound can be seen as a dual of this natural algorithm. To see this, we describe what happens when we apply the lower bound analysis to the algorithm. We argue informally. Consider two intervals of 2^k operations. The information transfer between the intervals is associated with a node of height k in the lower-bound tree. On the other hand, the indices of the operations will form a relatively uniformly spaced set of $O(2^k)$ indices. Thus, the distance in index space between a query from the right interval and the closest update from the left interval will usually be around $n/2^k$. The algorithm's tree passes information between the update and the query through the lowest common ancestor of the two indices. Because of the separation between the indices, this will usually be a node at level around $\lg n - k$. Thus, we can say that our lower bound is roughly an upside-down view of the upper bound. The information passed through the k^{th} level from the bottom of one tree is roughly associated with the k^{th} level from the top of the other tree.

6. Handling Queries with Low Output Entropy. The lower bound technique as presented so far depends crucially on the query answers having high entropy: the information transfer through a node is bounded from below by the entropy of all queries from the right subtree of the node. However, in order to prove lower bounds for dynamic language membership problems (such as dynamic connectivity), we need to be able to handle queries with binary answers. To prove lower bounds for a pair of adjacent intervals, it is tempting to consider the communication complexity between a party holding the updates from the left interval, and a party holding the queries from the right interval. Many bounds for communication complexity hold even for decision problems, so queries with binary output should not be a problem. However, a solution for the data structure does not really translate well into the communication-complexity setting. The query algorithm probes many cells, only a few of which (a logarithmic fraction) are in the left interval. If the party with the right interval communicates all these addresses, just to get back the answer "not written in the left interval" for most of them, the communication complexity blows up considerably. One could also imagine a solution based on approximate dictionaries, where the party holding the left interval sends a sketch of the cells that were written, allowing the other party to eliminate most of the uninteresting cell probes. However, classic lower bounds for approximate dictionaries [CFG$^+$78] show that it is impossible to send a sketch that is small enough for our purposes. The solution developed in this section is not based on communication complexity, although it can be rephrased in terms of nondeterministic communication complexity. While this solution is not particularly hard, we find it to be quite subtle.

6.1. Setup for the Lower Bound. Our approach is to construct hard sequences of operations that will have a fixed response, and the data structure need only confirm that the answer is correct. Such predictable answers do not trivialize the problem: the data structure has no guarantee about the sequence of operations, and the information it gathers during a query (by probing certain cells) must provide a certificate that the predicted answer is correct. In other words, the probed cells must uniquely identify the answer to the query, and thus must encode sufficient information to do so. As a consequence, our lower bounds hold even if the algorithm makes nondeterministic cell probes, or if an all-powerful prover reveals a minimal set of cells sufficient to show that a certain answer to a query is correct.

The machinery developed in this section is also necessary in the case of the partial-sums problem, if we want a lower bound for sequences of UPDATE and SELECT operations. Even though SELECT returns an index in the array, i.e., $\lg n$ bits, it is not clear how more than one bit of information can be used for a lower-bound argument. Instead, we consider a VERIFY-SUM operation, which is given a sum Σ and an index i, and tests whether the partial sum up to i is equal to Σ. In principle, this operation can be implemented by two calls to SELECT, namely by testing that $i = \text{SELECT}(\Sigma) = \text{SELECT}(\Sigma - 1) + 1$.

Below we give a single lower-bound proof that applies to both the partial-sums problem with verify, and dynamic connectivity. We accomplish this by giving a proof for the partial-sums problem over any group G with at least 2^δ elements, and then specializing G for the two problems we consider.

For the partial-sums problem with SELECT, we use $G = \mathbb{Z}/2^\delta\mathbb{Z}$. This introduces a slight complication, because VERIFY-SUM in modulo arithmetic can no longer be implemented by a constant number of calls to SELECT. To work around this issue, remember that our lower bound for VERIFY-SUM also holds for nondeterministic computation. To implement VERIFY-SUM(i, Σ) nondeterministically, we guess a b-bit quantity Σ' such that $\Sigma' \bmod 2^\delta = \Sigma$, and verify the old condition $i = \text{SELECT}(\Sigma') = \text{SELECT}(\Sigma' - 1) + 1$. We have implicitly assumed that SELECT is deterministic, which is natural because SELECT does not return a binary answer. Note that only one thread accepts, so there is no problem if SELECT updates memory cells (the updates made by the sole accepting thread are the ones that matter).

For the dynamic-connectivity problem, we use $G = S_{\sqrt{n}}$, i.e., the permutation group on $\sqrt{n}$ elements. Notice that now we have $\delta = \sqrt{n} \lg \sqrt{n} - \Theta(\sqrt{n})$, a very large quantity, unlike in the partial-sums problem where it was implied that $\delta < b$. Our proof never actually assumes any particular relation between δ and b.

To understand the relation between this problem and dynamic connectivity, refer to Figure 6.1. We consider a graph whose vertices form an integer grid of size $\sqrt{n}$ by $\sqrt{n}$. Edges only connect vertices from adjacent columns. Each vertex is incident to at most two edges, one edge connecting to a vertex in the previous column and one edge connecting to a vertex in the next column. These edges do not exist only when they cannot because the vertex is in the first or last column. The edges between two adjacent columns of vertices thus form a perfect matching in the complete bipartite graph $K_{\sqrt{n},\sqrt{n}}$, describing a permutation of order $\sqrt{n}$. More precisely, point (x, y_1) in the grid is connected to point $(x + 1, y_2)$ exactly when $\pi_x(y_1) = y_2$ for a permutation π_x. Another way to look at the graph is in terms of permutation networks. We can imagine that the graph is formed by $\sqrt{n}$ horizontal wires, going between permutation boxes. Inside each box, the order of all wires is changed arbitrarily.

Our graph is always the disjoint union of $\sqrt{n}$ paths. This property immediately implies that the graph is plane, because any embedding maintains planarity (though the edges may have to be routed along paths with several bends).

The operations required by the partial-sums problem need to be implemented in terms of many elementary operations, so they are are actually "macro-operations". Macro-operations are of two types, UPDATE and VERIFY-SUM, and all receive as parameters a permutation and the index x of a permutation box. To perform an update, all the edges inside the named permutation box are first deleted, and then reconstructed according to the new permutation. This translates to $\sqrt{n}$ DELETE's and $\sqrt{n}$ INSERT's in the dynamic connectivity world. Queries on box x test that point $(1, y)$ is connected to point $(x+1, \pi(y))$, for all $y \in \{1, 2, \ldots, \sqrt{n}\}$. This requires $\sqrt{n}$ connectivity queries. The conjunction of these tests is equivalent to testing that the composition of $\pi_1, \pi_2, \ldots, \pi_x$ (the permutations describing the boxes to the left) is identical to the

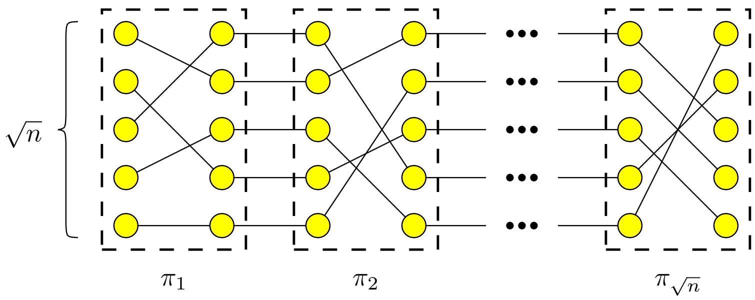

FIG. 6.1. *Our graphs can be viewed as a sequence of permutation boxes (dashed). The horizontal edges between boxes are in fact contracted in the actual graphs.*

given permutation π — the VERIFY-SUM in the partial-sums world.

As stated before, the lower bound we obtain is $\Omega(\frac{\delta}{b}\lg n)$. For dynamic connectivity, we are interested in $b = \Theta(\lg n)$, which is the natural word size for this problem. As we saw already, $\delta = \Theta(\sqrt{n}\lg n)$. Thus, our lower bound translates into $\Omega(\sqrt{n}\lg n)$. This is a lower bound for the macro-operations, though, which are implemented through $O(\sqrt{n})$ elementary operations. Therefore, the lower bound for dynamic connectivity is $\Omega(\lg n)$, as desired. The same calculation applies to the trade-off expressions, which essentially means that the $\frac{\delta}{b}$ term should be dropped to obtain the true bound for dynamic connectivity.

6.2. Proof of the Lower Bound. As before, the sequence of operations alternates between UPDATE or VERIFY-SUM. The index queried or updated is chosen uniformly at random. If the operation is an UPDATE, we select a random element of G for the value Δ. If the operation is VERIFY-SUM, we give it the composition of the elements before the queried index. This means that the data structure will be asked to prove a tautology, involving the partial sum up to that index.

Because of this construction of the hard sequence, at least one nondeterministic thread for each query should accept. For every random input, let us fix one accepting thread for each operation. When we mention cells that are "read", we mean cells read in this chosen execution path; by definition of the model, writes are the same for all accepting threads. As in the framework discussion, we are interested in lower bounds for the information transfer between adjacent intervals of operations. The following lemma is an analog of Lemma 5.1.

LEMMA 6.1. *Consider two adjacent intervals of operations such that the left interval contains L updates, the right intervals contains L queries, and overall the intervals contain $O(\sqrt[3]{n})$ operations. Let w be the number of write instructions executed during the first interval, and let r be the number of read instructions executed during the second interval. Also let c be the number of read instructions executed during the second interval that read cells last written during the first interval. Then $E[c] = \Omega\left(L\frac{\delta}{b}\right) - O\left(\frac{E[s]}{b}\right)$, where $s = \lg\binom{r+w}{r}$.*

Note that the lower bound of this lemma is weaker than that of Lemma 5.1, because of the additional term $\frac{E[s]}{b}$. Before we prove this lemma (in the next section), let us see that this term ends up being inconsequential, and we can still prove the same bounds and trade-offs.

Consider a sequence of $m = \Theta(\sqrt[3]{n})$ operations, and let T be the total running time of the data structure. We construct a complete binary tree over these operations.

Consider a node v that is a right child, and let L be the number of leaves in its subtree. By Lemma 6.1, we have $E[c] = \Omega\left(L\frac{\delta}{b}\right) - O\left(\frac{E[s]}{b}\right)$, where c is the number of cell probes associated with v. Note that $s = \lg\binom{r+w}{r} \le r + w$; thus, s is bounded by the number of read instructions in v's subtree, plus the number of write instructions in the subtree of v's left sibling. Summing for all nodes on a level, s counts all read and write instructions at most once, so we obtain $E[\sum c_i] = \Omega\left(m \cdot \frac{\delta}{b}\right) - O\left(\frac{E[T]}{b}\right)$. We sum up the lower bounds for each level to obtain a lower bound on $E[T]$. We obtain that $E[T] = \Omega\left(m \cdot \frac{\delta}{b}\lg n\right) - O\left(\lg n \cdot \frac{E[T]}{b}\right)$. Because $\lg n \le b$, this means that $E[T] = \Omega\left(m \cdot \frac{\delta}{b}\lg n\right)$. This result implies an average-case amortized lower bound per operation of $\Omega\left(\frac{\delta}{b}\lg n\right)$.

To obtain trade-off lower bounds, we apply the same reasoning as in Section 5.5. The only thing we have to do is verify that the new term depending on $E[s]$ does not affect the end result. When we sum the lower bounds for one level in the tree, we lose a term of $O\left(\sum \frac{E[s_i]}{b}\right)$ compared to the old bound. Here i ranges over all nodes at that level. We must understand $\sum s_i = \lg \prod \binom{r_i+w_i}{r_i}$ in terms of T, the total running time for the entire sequence of operations.

The quantity $\prod \binom{r_i+w_i}{r_i}$ counts the total numbers of ways to choose r_i elements from a set $r_i + w_i$, where we have a different set for each i. This is bounded from above by the number of ways to choose $\sum r_i$ elements from a single set of $\sum(r_i + w_i)$ objects. Thus, $\sum s_i \le \lg\binom{\sum(r_i+w_i)}{\sum r_i}$. Assume we have upper bounds $\sum r_i \le U_r$ and $\sum w_i \le U_w$. Then, we can write $\binom{\sum(r_i+w_i)}{\sum r_i} \le \binom{U_r+U_w}{\sum r_i} \le \binom{2(U_r+U_w)}{\sum r_i} \le \binom{2(U_r+U_w)}{U_r}$. The last inequality holds because $\binom{n}{k}$ increases with k for $k \le \frac{n}{2}$. We entered this regime by artificially doubling $U_r + U_w$. Because r_i and w_i are symmetric, we also have $\binom{\sum(r_i+w_i)}{\sum r_i} = \binom{\sum(r_i+w_i)}{\sum w_i} \le \binom{2(U_r+U_w)}{U_r}$.

Now we need to develop the upper bounds U_r and U_w. For the case $t_u \le t_q$, our proof considered intervals formed by a node and all its left siblings. Thus, $\sum r_i$ counts each read instruction once, for the node it is under; so $\sum r_i \le T$. On the other hand, $\sum w_i \le B \cdot T$, because a write instruction is counted for every right sibling of its ancestor on the current level. For the case $t_u > t_q$, we consider intervals formed by a node, and all its right siblings. Thus, $\sum w_i \le T$ and $\sum r_i \le B \cdot T$.

Using these bounds, we see that $\sum s_i \le \lg\binom{2(B+1)T}{T} = O(T \lg B)$. Because this upper bound holds in any random instance, it also holds in expectation: $E[\sum s_i] = O(E[T]\lg B)$. So our lower bound loses $O\left(\frac{E[T]\lg B}{b}\right)$ per level, which, over all levels, sums to $O\left(\frac{E[T]\lg B}{b}\log_B n\right) = O\left(E[T]\frac{\lg n}{b}\right)$. Because $\lg n \le b$, our lower bound on $E[T]$ is equal to the old lower bound minus $O(E[T])$. Thus, we lose only a constant factor in the lower bound, and the results of Section 5.5 continue to hold.

6.3. Proof of Lemma 6.1. The proof is an encoding argument, which is similar in spirit to the proof of Lemma 5.1, but requires a few significant new ideas. The difference from the previous proof is that the partial sums that we want to encode are no longer returned by queries, but rather they are given as parameters. Our strategy is to recover the partial sums by simulating each query for all possible parameters, and see which one leads to an accept. However, these simulations may read a large number of cells, which we cannot afford in the encoding. Instead, we add a new part to the encoding which enables us to stop simulations that try to read cells we don't know. The difficulty is making this new component of the encoding small enough.

As before, we consider two adjacent intervals of operations, the first spanning $[i, j-1]$ and the second $[j, k]$. We propose an encoding for the partial sums passed to the `VERIFY-SUM` operations during $[j, k]$, given the variable G defined in Lemma 5.3. By this lemma, such an encoding must have size $\Omega(L\delta)$ bits.

The encoder first simulates the entire execution of the data structure. Each query is given the correct partial sum, so it must accept. We choose arbitrarily one of the accepting threads. Consider the following sets of cells, based on this computation history:

$W =$ cells which are updated by the data structure during the interval of time $[i, j-1]$, and never read during $[j, k]$.

$R =$ cells which are read by the data structure during $[j, k]$ and their last update before the read happened before time i.

$C =$ cells which are read by the data structure during $[j, k]$ and their last update before the read happened during $[i, j-1]$.

These are simple sets, so, for example, cells written multiple times during $[i, j-1]$ are only included once in W. We have $|C| = c, |W| \leq w, |R| \leq r$. Note that all of $c, |W|, w, |R|$ and r are random variables, because the data structure can behave differently depending on the Δ's passed to the updates. We will give an encoding for the queried partial sums that uses $O(b) + c \cdot 2b + O(s)$ bits, where $s = \lg \binom{r+w}{r}$. Because the expected size of our encoding must be $\Omega(L\delta)$, we obtain that $E[c] + \frac{E[s]}{\Theta(b)} = \Omega\left(L\frac{\delta}{b}\right)$ and therefore $E[c] = \Omega\left(L\frac{\delta}{b}\right) - O\left(\frac{E[s]}{b}\right)$.

Our encoding consists of two parts. The first encodes all information about the interesting cell probes (the information transfer): for each cell in C, we encode the address of the cell and its contents at time j. This uses $O(b)$ bits to write the size of C, and $c \cdot 2b$ for the information about the cells. The second part is concerned with the "uninteresting" cell probes, i.e., those in R. This accounts for a covert information transfer: the fact that a cell was *not* written during $[i, j-1]$ is a type of information transmitted to $[j, k]$. The part certifies that W and R are disjoint, by encoding a set S, such that $R \subset S$ and $W \subset \overline{S}$. We call S a separator between R and W. To efficiently encode a separator, we need the following result:

LEMMA 6.2. *For any integers a, b, u with $a + b \leq u$, there exists a system of sets $\mathbb{S}$ with $\lg |\mathbb{S}| = O(\lg\lg u + \lg \binom{a+b}{a})$ such that, for all $A, B \subset \{1, 2, \ldots, u\}$ with $|A| \leq a, |B| \leq b, A \cap B = \emptyset$, there exists an $S \in \mathbb{S}$ satisfying $A \subset S$ and $B \subset \overline{S}$.*

Proof. It suffices to prove the lemma for $|A| = a$ and $|B| = b$, because we can simply add some elements from $\{1, 2, \ldots, u\} \setminus (A \cup B)$ to pad the sets to the right size. We use the probabilistic method to show that a good set system exists. Select a set S randomly, by letting every element $x \in \{1, 2, \ldots, u\}$ be in the set with probability $p = \frac{a}{a+b}$. Then, for any pair A, B, the probability that $A \subset S$ and $B \subset \overline{S}$ is $p^a(1-p)^b$. The system $\mathbb{S}$ will be formed of sets chosen independently at random, so the probability that there is no good S for some A and B is $\left(1 - p^a(1-p)^b\right)^{|\mathbb{S}|} \leq \exp\left(-p^a(1-p)^b|\mathbb{S}|\right)$. The number of choices for A and B is $\binom{u}{a}\binom{u-a}{b} \leq u^{a+b}$. So the probability that there is no good set in $\mathbb{S}$ for any A, B is at most $u^{a+b}\exp\left(-p^a(1-p)^b|\mathbb{S}|\right) = \exp\left((a+b)\ln u - p^a(1-p)^b|\mathbb{S}|\right)$. As long as this probability is less than 1, such a system $\mathbb{S}$ exists. So we want $(a+b)\ln u < p^a(1-p)^b|\mathbb{S}| = \left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b|\mathbb{S}|$. We want to choose a system of size greater than $\frac{(a+b)^{a+b+1}\ln u}{a^a b^b}$. Then $\lg|\mathbb{S}| = \Theta((a+b+1)\log_2(a+b) + \lg\lg u - a\log_2 a - b\log_2 b)$. Assume by symmetry that $a \leq b$. Then $\lg|\mathbb{S}| = \Theta\left(\lg\lg u + a\lg\frac{b}{a} + a \cdot \frac{b}{a}\log_2\left(1 + \frac{a}{b}\right)\right)$.

Let $t = \frac{b}{a}$; then $t \log_2(1 + \frac{1}{t}) = \log_2\big((1+\frac{1}{t})^t\big) \to \log_2 e$ as $t \to \infty$. We then have $\frac{b}{a}\log_2\big(1+\frac{a}{b}\big) = \Theta(1)$, so our result simplifies to $\lg|\mathbb{S}| = \Theta(\lg\lg u + a\lg(b/a))$. It is well known that $a\lg(b/a) = \Theta(\lg\binom{a+b}{a})$, for $a \le b$. □

We apply this lemma with parameters r, w and 2^b. First, we encode r and w, using $O(b)$ bits. The encoding and decoding algorithms can simply iterate over all possible systems $\mathbb{S}$ for the given r and w, and choose the first good one (in the sense of the lemma). Given this unique choice of a system, a separator between R and W is the index of an appropriate set in the system. This index will occupy $O(\lg\lg(2^b) + \lg\binom{r+w}{r}) = O(\lg b + s)$ bits.

It remains to show that this information is enough to encode the sequence of queried partial sums. We simulate the data structure for the interval $[j, k]$, and prove by induction on time steps that all cell writes made by these operations are correctly determined, and all partial sums appearing in `VERIFY-SUM`'s are recovered. Updates are easy to handle, because their parameters are known given G, and they are deterministic. Thus, we can simply simulate the update algorithm. We are guaranteed that all cells that are read and have a chronogram in $[i, j-1]$ appear in C, so we can identify these cells and recover their contents. All other cells have a known content given G, so we can correctly simulate the update.

In the case of `VERIFY-SUM`, we do not actually know the sum passed to it, so we cannot simply simulate the algorithm. Instead, we try all partial sums that could be passed to the query, and for each one try all possible execution paths that the data structure can explore through nondeterminism. The cell probes made while simulating such a thread fall in one of the following cases:

- the cell was written by the data structure after time j. This case can be identified by looking at the set of cells written during the simulation. By the induction hypothesis, we have correctly determined the cell's contents.
- the cell is in C. We recover the contents from the encoding.
- the cell is on R's side of the separator between R and W. Then, it was not written during $[i, j-1]$, and thus it has the old value before time i. Given G, everything is fixed before time i, so we know the cell's contents.
- the cell is on W's side of the separator. Then this thread of execution cannot be in the computation history chosen by the encoding algorithm. We abort the thread.

For each query, there exists a unique partial sum for which is should accept. Furthermore, one accepting thread is included in the computation history of the encoder. Thus, we identify at least one thread which accepts and is not aborted because of the last case above. Because the data structure is correct, all accepting threads must be for the same partial sum, so we correctly identify the sum. By definition of the nondeterministic model, the cell writes are identical for all accepting threads, so we correctly determine the cell writes, as well.

It should be noted that, even though the size of the encoding only depends on the characteristics of one accepting thread per query, the separator allows us to handle an arbitrary number of rejecting threads. All such threads (including all threads for incorrect partial sums) are either simulated until they reject, or they are aborted.

6.4. Handling Monte Carlo Randomization. This section shows that the logarithmic lower bound for dynamic connectivity is also true if we allow Monte Carlo randomization with two-sided error probability at most n^{-c}, for constant c (that is, the data structure must be correct with high probability). The idea is to make the decoding algorithm from the previous section use only a polynomial number of calls to data structure operations.

Assume for now that the data structure is deterministic. The previous decoding algorithm simulates a large number of primitive connectivity operations for every query. A partial sum is recovered by simulating VERIFY-SUM for all possible sums. Remember that a partial sum in the dynamic connectivity problem is a permutation in $S_{\sqrt{n}}$, obtained by composition of the permutations up to a certain column k. Thus, there are $(\sqrt{n})!$ partial sums to try – a huge quantity. However, we can recover the partial-sum permutation by simulating at most $(\sqrt{n})^2$ CONNECTED queries: it suffices to test connectivity of every point in the first column with every point on the k^{th} column. First, the decoder simulates CONNECTED queries between the first node in column one, and every node in column k. Exactly one of these queries was executed by the encoder, so that query should accept. The other queries will reject or be aborted. Now the writes made by the accepting query are incorporated in the data structure. The decoder continues to simulate query calls between the second node in column one, and all nodes in column k, and so on.

Now assume that the data structure makes an error with probability at most n^{-c}, for a sufficiently large constant c. We make the encoding randomized; the random bits are those used to initialize the memory of the data structure. We assume both the encoder and the decoder receive the same random bits, so they can both simulate the same behavior of the data structure. By the minimax principle, we can fix that random bits if we are only interested in the expected size of the encoding for a known input distribution (which is the case).

The decoding algorithm described above will work if all correct queries accept, *and* all incorrect queries would reject if they were executed instead of the correct one. We can simulate the execution of any query, or abort it only if it is not one of the correct queries. So if all incorrect queries reject, their simulation will either reject or be aborted. Because we only consider polynomial sequences of operations, we simulate at most poly(n) queries (including the incorrect ones). The probability that any of them will fail is at most $n^{-c'}$, for some arbitrarily large constant c' (depending of c). Because the decoder has the same coins as the encoder, the encoder can predict whether the decoder will fail. Thus, it can simply add one bit saying whether the old encoding is used (when the decoder works), or the entire input is simply included in the encoding (if the old decoder would fail). The expected size of the encoding grows by at most $1 + n^{-c'} \cdot \text{poly}(n) < 2$ for sufficiently large c'. So the bounds of Lemma 6.1 remain the same. Then, the bounds and trade-offs derived for dynamic connectivity hold even if the data structure answers correctly with high probability.

7. Handling a Higher Word Size. For the partial-sums problem, it is natural and traditional to consider the case $\delta = o(b)$. For ease of notation, we will let $B = \frac{b}{\delta}$. For dynamic connectivity, our motivation comes from external-memory models. For this problem, a "memory cell" is actually an entire page, because that is the unit of memory that can be accessed in constant time. In this case, B is what is usually referred to as "page size"; the number of bits in a page is $b = B \cdot \lg n$. For both problems, the lower bound we obtain is $\Omega(\log_B n)$.

We note that the analysis from the previous sections gives a tight bound on the number of bits that must be communicated: $\Omega(\delta \lg n)$. Given that we can pack b bits in a word, it is straightforward to conclude that $\Omega(\frac{\delta}{b} \lg n) = \Omega(\frac{\lg n}{B})$ read instructions must be performed. Our strategy for achieving $\Omega(\frac{\lg n}{\lg B})$ is to argue that an algorithm cannot make efficient use of all b bits of a word, if future queries are sufficiently unpredictable. Intuitively speaking, if we need δ bits of information from a certain time epoch to answer a query, and there are $t \cdot \frac{b}{\delta}$ possible future queries that would also need δ bits of information from the same epoch ($t > 1$), a cell probe cannot be very effective. No matter what information the cell probe gathers, we have a probability of at most $1/t$ that it has gathered all the information necessary for a random future query, so with constant probability the future query will need another cell probe. The reader will recognize the similarity, at an intuitive level, with the round-elimination lemma from communication complexity [MNSW98, SV]. Also note that our proof strategy hopelessly fails with any deterministic sequence of indices, such as the bit-reversal permutation. Thus, we are identifying another type of hardness hidden in our problems.

Unfortunately, there are two issues that complicate our lower bounds. The first is that, for dynamic connectivity, we need to go beyond the VERIFY-SUM abstraction, and deal with CONNECTED queries directly. To see why, remember that a VERIFY-SUM macro-query accesses a lot of information ($\Theta(\sqrt{n} \lg n)$ bits) in a very predictable fashion, depending on just one query parameter. Thus, we do not have the unpredictability needed by our lower bound. The second complication is that, for the partial-sums problem, we can handle VERIFY-SUM only when $\delta = \Omega(b)$. When $\delta = o(\lg n)$, the information per query is not enough to hide the cost of the separators from Lemma 6.2. However, we can still obtain lower bounds for SUM and SELECT, without nondeterminism, using a rather simple hack.

In Section 7.1, we describe a new analysis for adjacent intervals of operations, which is the gist of our new lower bounds. In Section 7.2, we show how this new lower bound can be used for the partial-sums problems, whereas in Section 7.3, we show how to apply it to dynamic connectivity.

7.1. A New Lower Bound for Adjacent Intervals. We now consider an abstract data-structure problem with two operations, UPDATE and QUERY. We do not specify what UPDATE does, except that it receives some parameters, and behaves deterministically based on those. A query receives two parameters i and q, and returns a boolean answer. We refer to i as an index. For any admissible i, there exists a unique q which makes the query accept. The parameter q is a δ-bit value, where δ is a parameter of the problem; we let $B = b/\delta$. The implementation of QUERY can be nondeterministic. We assume that the hard instance of the problem comes from some random distribution, but that the pattern of updates and queries is deterministic. In the hard instance, each QUERY receives the q which makes it accept. We will assume that the random features of each operation are chosen by the distribution independently of the choices for other operations. Though we do not really need this assumption, it is true in both problems we consider, and assuming independence simplifies exposition.

Now consider two intervals of operations $[i, j-1]$ and $[j, k]$, and let L be the number of queries in the second interval. We make an information-theoretic assumption, which we will later prove for both the partial-sums and dynamic-connectivity problems. To describe this assumption, pick a random $t \in [j, k]$ such that the t^{th} operation is a query. Also pick a set Q of BL random queries that could have been generated as query t. Now, imagine simulating each such query starting with the state of the data structure at time $t - 1$. Our assumption is essentially that the correct q's for all original queries from $[j, k]$, plus the simulated queries in Q, have high entropy. More specifically, let Z be the random variable specifying all updates outside $[i, j-1]$ and the indices for all queries, including those in Q. We assume that the vector of q's has entropy $\Omega(BL\delta)$ given Z.

Let w be the number of write instructions executed during the first interval, and let r be the number of read instructions executed during the second interval. Also let c be the number of read instructions executed during the second interval that read cells last written during the first interval. Under the assumptions above, we prove $E[c] = \Omega(L) - O(\frac{s}{b})$, where $s = (B \cdot E[r]) \log_2 \left(1 + \frac{E[w]}{B \cdot E[r]}\right)$.

We now outline the proof strategy. The probability that query t reads at least one cell from $[i, j-1]$ is at most $\frac{E[c]}{L}$. If $E[c]$ were small, so would this probability. That would mean that a large fraction of the queries from Q (random queries that could be executed at time t) would not need to read any cell written during $[i, j]$. On the other hand, the queries recover $\Omega(BL\delta) = \Omega(Lb)$ bits of information about the updates in the left interval. Because most queries don't need to read another cell, most of this information must have already been recovered by the cell probes made in $[j, t-1]$. There are at most $E[c]$ probes in expectation, each reading b bits, so the recovered information is not enough when $E[c]$ is small.

Encoding Algorithm. As the first step of the formal proof, we describe the algorithm encoding the correct q's. First simulate the entire execution of the data structure, with the real query at time t. For each query, include an arbitrary accepting thread in the computation history. Based on this computation history, consider the following sets:

$C =$ cells that are written during $[i, j-1]$ and read during $[j, k]$.
$W =$ cells that are written during $[i, j-1]$ but not read during $[j, k]$.
$R_1 =$ cells that are read during $[j, k]$, but never written during $[i, j-1]$.

Now simulate the queries in Q starting from the state of the data structure at time $t-1$. As before, we only pass correct parameters to these queries. Call *easy queries* the queries for which there exists an accepting thread which does not read any cell in W; call such a thread a *good thread*. The rest of the queries are *hard queries*; let h be the number of hard queries. Let R_2 be the union of the cells read by a arbitrary good thread of every easy query, from which we exclude the cells in C. By definition of easy queries, R_2 is disjoint from W. Let $R = R_1 \cup R_2$; R is also disjoint from W.

The encoding has four parts:

1. encode c and for each cell in C, the address and contents of the cell;
2. a separator (as given by Lemma 6.2) between R and W;
3. encode h, and the set of hard queries. The set takes $\lg \binom{|Q|}{h} = \lg \binom{BL}{h}$ bits.
4. the correct q for each hard query, as an array of size h. This takes $h\delta$ bits.

The third part of the encoding could be avoided for our current problem, because the separator can be used to recognize hard queries. However, we will later consider a variation in which we discard the separator, and then encoding which queries are hard could no longer be avoided.

Decoding Algorithm. We now describe how to recover the correct q's given Z and the previous encoding. By definition, a separator of R and W is also a separator for R_1 and W. Given this separator and complete information about C, we can simulate the real operations in the second interval, as argued in Lemma 6.1, and recover their correct q's. Now we have to recover the correct parameters for the queries in Q. For hard queries, this is included in the encoding. For each easy query, each possible q, and all threads, we try to simulate the thread starting with what we know about the data structure at time $t-1$. Each cell that is probed falls in one the following cases:

- the cell was written during $[j, t-1]$. Because we simulated the data structure in this interval, we can identify this condition and recover the cell contents.
- the cell is in C. We recover the contents from the encoding.
- the cell is on R's side of the separator between R and W. Then, it was not written during $[i, j-1]$, and we can recover the cell contents because Z includes perfect information before time i.
- the cell is on W's side of the separator. Then, this thread of execution cannot be among the chosen good threads for the easy queries, so we abort it.

For the chosen good thread of an easy query, the encoder included its probes outside of C in the set R_2, so simulation of this thread is never aborted. Thus, for each easy query, we find at least one accepting thread, and recover the correct q.

Analysis. We want to bound the size of the separator. We have $|W| \le w$, $|R_1| \le r$, so it remains to bound $|R_2|$. In expectation over a random t and a random choice of the queries in the second interval, the number of cells read by query t is at most $\frac{E[r]}{L}$. We are simulating a set of BL queries as if they happened at time t. In expectation, the total number of cell probes performed by these is at most $BL\frac{E[r]}{L} = B \cdot E[r]$, which also bounds $E[|R_2|]$. Then $E[|R|] \le E[|R_1|] + E[|R_2|] = O(B)E[r]$. To specify the separator, we need $O(b)$ bits to write $|W|$ and $|R|$, and then, by Lemma 6.2, $O\left(\lg b + \log_2 \binom{|W|+|R|}{|R|}\right)$ bits for the index into the system of separators. The total size is $O\left(b + |R| \log_2\left(1 + \frac{|W|}{|R|}\right)\right)$ bits. The function $(x, y) \mapsto x \log_2(1 + \frac{y}{x})$ is concave, so the expected size is upper bounded by moving expectations inside. Then, the expected size of the separator is $O\left(b + (B \cdot E[r]) \lg\left(1 + \frac{E[w]}{B \cdot E[r]}\right)\right) = O(b + s)$.

To analyze the rest of the encoding, we need to bound h. For a random t, the expected number of cell probes from the first interval that are made by query t is at most $\frac{E[c]}{L}$. This means that a random query at position t is bad with probability at most $\frac{E[c]}{L}$. Thus, $E[h] = BL\frac{E[c]}{L} = B \cdot E[c]$. Explicitly encoding the correct q's for the hard queries takes $E[h]\delta = b \cdot E[c]$ bits in expectation. This is the same as the space taken to encode the contents of cells in C. Encoding which queries are hard takes space $O(b) + \lg \binom{BL}{h} = O\left(b + h \lg \frac{BL}{h}\right)$. The function $x \to x \lg \frac{\gamma}{x}$ is concave for constant γ, so the expected size is at most $O\left(b + E[h] \lg \frac{BL}{E[h]}\right) = O\left(b + B \cdot E[c] \lg \frac{L}{E[c]}\right)$.

We have shown an upper bound of $O\left(E[c]b + s + B \cdot E[c] \lg \frac{L}{E[c]}\right)$ on the expected total size of the encoding. Let $\varepsilon > 0$ be an absolute constant to be determined. If $E[c] \ge \varepsilon L$, there is nothing to prove. Otherwise, observe that $x \mapsto x \log_2(2 + \frac{\gamma}{x})$ grows with x for constant γ, so the last term of the encoding size becomes $O\left(B\varepsilon L \lg \frac{1}{\varepsilon}\right)$. The assumed lower bound on the size of the encoding is $\Omega(BL\delta) = \Omega(bL)$, so we obtain $E[c] = \Omega(L) - O\left(\frac{\varepsilon}{\delta} L \lg \frac{1}{\varepsilon}\right) - O\left(\frac{s}{b}\right)$. Note that $\varepsilon \lg \frac{1}{\varepsilon}$ goes to zero as ε goes to zero. Then, assuming $\delta \ge 2$, there is an absolute constant ε such that the second term of the lower bound is a constant fraction of the first term. We thus obtain $E[c] = \Omega(L) - O\left(\frac{s}{b}\right)$.

Deterministic queries. Now we consider a variation of our original problem, in which queries are deterministic, and they return q, as opposed to verifying a given q. The only change in our analysis is that we do not need the separator. Indeed, each query can be simulated unambiguously, because it only receives a known index, and it is deterministic. Then, the separator term in our lower bound disappears, and we obtain $E[c] = \Omega(L)$.

7.2. The Partial-Sums Problem. Our hard instance is the same as in Section 5.5: we consider blocks of t_q random updates and t_u queries to random indices. We begin by showing a lower bound for two intervals based on the analysis from the previous section. Let c, r, w be as defined in the previous section.

LEMMA 7.1. *Consider two adjacent intervals of operations such that the left interval contains $B \cdot L$ updates, the right interval contains L queries, and overall the intervals contain $O(\sqrt[3]{n})$ operations. The following lower bounds hold:*

- *In the case of* SUM *queries, $E[c] = \Omega(L)$.*
- *In the case of* VERIFY-SUM *queries, $E[c] = \Omega(L) - O\left(\frac{s}{b}\right)$, where we define $s = (B \cdot E[r]) \log_2 \left(1 + \frac{E[w]}{B \cdot E[r]}\right)$.*

Proof. This follows from the analysis in the previous section, as long as we can show the information-theoretic assumption made there. Specifically, we pick a query from the second interval, and imagine simulating BL random queries in its place. We need to show that the partial sums of the original queries and these virtual queries have entropy $\Omega(BL\delta)$, given the indices of all queries, and the indices and Δ values for all queries outside the left interval (the variable Z from the previous section). To prove this, we apply Lemma 5.3. Because that lemma only deals with the partial sums (a feature of the problem instance) and not with computation, it doesn't matter that we are simulating the BL queries at the same time. The partial sums would be the same if the queries were ran consecutively. Then, the lemma applies, and shows our entropy lower bound. Note that the variable G in Lemma 5.3 describes all queries, including the simulated ones (which the lemma thinks are consecutive). This is exactly the variable Z. □

We now show how to use this lemma to derive our lower bounds. Our analysis is similar to that of Section 6.2, with two small exceptions. The first is that there is an inherent asymmetry between the left and right interval in Lemma 7.1. Because of this, we can only handle the case $t_q = O(t_u)$. The second change is that the definition of s is somewhat different from that in Lemma 6.1; roughly, s is larger because $E[r]$ is multiplied by B. We will show lower bounds of the form $t_q \left(\lg B + \lg \frac{t_u}{t_q}\right) = \Omega(\lg n)$.

We consider a balanced tree with branching factor $\beta = 2B\frac{t_u}{t_q}$, over $m = \Theta(\sqrt[6]{n})$ blocks. Because for $\max\{t_q, t_u\} = \Omega(\sqrt[6]{n})$, our trade-off states $\min\{t_q, t_u\} = \Omega(1)$, we may assume $t_u + t_q = O(\sqrt[6]{n})$. Then there are $O(\sqrt[3]{n})$ operations in total, as needed. We will consider right intervals formed by a node of the tree, and left intervals formed by all its left siblings. The choice of β gives the right proportion of updates in the left interval compared to queries in the right interval, for any node which is in the right half of its siblings. Then, we can apply Lemma 7.1.

First, consider the case of SUM queries, so there is no term depending on s. Note that the $\Omega(L)$ term is linear in the number of queries, so summing it up over the entire tree yields a lower bound on the total time of $E[T] = \Omega(m t_u \log_\beta n)$. By the definition of the blocks, $E[T] = m \cdot 2t_u t_q$, so $t_q = \Omega(\log_\beta n)$, which is equivalent to $t_q \left(\lg B + \lg \frac{t_u}{t_q}\right) = \Omega(\lg n)$.

Now we consider nondetermistic VERIFY-SUM queries, assuming $\delta = \Omega(\lg n)$. There is a new term in the lower bound on $E[T]$, given by the sum of the s terms over all nodes. First consider the sum for all nodes on one level: $\sum s_i = \sum (B \cdot E[r_i]) \log_2 \left(1 + \frac{E[w_i]}{B \cdot E[r_i]}\right)$. We have $\sum r_i \le T$, because each read is counted for the node it is under, and $\sum w_i \le \beta T$, because each write is counted for the siblings of the node it is under. These inequalities must also hold in expectation, so $\sum B \cdot E[r_i] \le$

$B \cdot E[T]$ and $\sum E[w_i] \le \beta E[T]$. Because the function $(x, y) \mapsto x \log_2(1 + \frac{y}{x})$ is concave, $\sum s_i$ is maximized when $E[r_i]$ and $E[w_i]$ are equal. Then $\sum s_i \le B \cdot E[T] \log_2 \left(1 + \frac{\beta E[T]}{B \cdot E[T]}\right) = O(E[T] B \lg(t_u/t_q))$. When we sum this over $O(\log_\beta n)$ levels, we obtain $E[T] \cdot O\left(B \lg(t_u/t_q) \frac{\lg n}{\lg(B t_u/t_q)}\right) = E[T] \cdot O(B \lg n)$.

Thus, our overall lower bound becomes $E[T] = \Omega(m t_u \log_\beta n) - O(\frac{1}{b} E[T] B \lg n)$. Expanding B, $E[T] = \Omega(m t_u \log_\beta n) - O(E[T] \frac{\lg n}{\delta})$. For $\delta = \Omega(\lg n)$, we obtain $E[T] = \Omega(m t_u \log_\beta n)$, which is the same lower bound as for SUM queries. This bound holds for VERIFY-SUM queries, even with nondeterminism, and, as shown in Section 6.1, also for SELECT queries.

For the case $\delta = O(\lg n)$, we can obtain the same lower bound on SELECT by a relatively simple trick. This means that the trade-off lower bound for SELECT holds for any δ, though we cannot prove it in general for VERIFY-SUM. The trick is to observe that for small δ (e.g. $\delta < \frac{1}{3} \lg n$), we can stretch (polynomially) an instance of SUM into an instance of SELECT. Because we already have a lower bound for SUM, a lower bound for SELECT follows by reduction.

Consider the SUM problem on an array $A[0 \mathinner{..} \sqrt[3]{n} - 1]$, where each element has δ bits. This implies $0 \le A[i] < \sqrt[3]{n}$, and any partial sum is less than $n^{2/3}$. Now we embed A into an array $A'[0 \mathinner{..} n-1]$ by $A'[i \cdot n^{2/3}] = A[i]$. The $n^{2/3} - 1$ spacing positions between elements from A are set to 1 in the initialization phase, and never changed later. An update in A translates into an update in A' in the appropriate position. Now assume we want to find $\sigma = \sum_{i=0}^{k} A[i]$. We run SELECT$((k+1)(n^{2/3} - 1))$ in A'. We have $\sum_{j=0}^{t + k \cdot n^{2/3}} A[j] = t + \sigma + k(n^{2/3} - 1)$, for any $t < n^{2/3}$. Then, if SELECT returns $t + k \cdot n^{2/3}$, we know that $t + \sigma = n^{2/3} - 1$, so we find σ.

7.3. Dynamic Connectivity. The graph used in the hard sequence is the same as the one before (Figure 6.1): $\sqrt{n}$ permutation boxes, each permuting $\sqrt{n}$ "wires". Let t_u be the running time of an update (edge insertion or deletion), and t_q the running time of a query. We only handle $t_q \le t_u$. Our hard sequence consists of blocks of operations. Each block begins with a macro-update: for an index k (chosen as described below), remove all edges in the k^{th} permutation box, and insert edges for a random permutation. Then, the block contains $\frac{t_u}{t_q}\sqrt{n}$ CONNECTED queries. Each query picks a random node in the first column and a random index k, and calls CONNECTED on the node in the first column and the node on the k^{th} column which is on the same path. This means that all queries should be answered in the affirmative; the information is contained in the choice of the node from the k^{th} column.

We still have to specify the sequence of indices of the macro-updates. We use a deterministic sequence to ensure that updates which occur close in time touch distant indices. This significantly simplifies the information-theoretic analysis. Our hard sequence consists of exactly $\sqrt{n}$ block. Each macro-update touches a different permutation box; the order of the boxes is given by the bit-reversal permutation (see Section 5.6) of order $\sqrt{n}$. Now consider a set of indices $S = \{i_1, i_2, \dots\}$ sorted by increasing i_j. We say S is *uniformly spaced* if $i_{j+1} - i_j = \sqrt{n}/(|S| - 1)$ for every j.

LEMMA 7.2. *Consider two adjacent intervals of operations, such that the second one contains L queries, and the indices updated in the first interval contain a uniformly-spaced subset of cardinality* $\Theta(BL/\sqrt{n})$. *Then* $E[c] = \Omega(L) - O\left(\frac{s}{b}\right)$, *where* $s = B \cdot E[r] \lg\left(1 + \frac{E[w]}{B \cdot E[r]}\right)$.

Proof. This lemma follows from Section 7.1, if we show the information-theoretic assumption used there. For our problem, $\delta = \Theta(\lg n)$. Imagine picking a random query from the right interval and simulating BL random queries in its place. The variable Z denotes the random choices for all queries, and for updates outside the left interval. We need to show that the entropy of the correct parameters for all queries in the right interval, including the simulated ones, given Z, is $\Omega(BL \lg n)$.

Remember that all updates are to different boxes, so an update is never overwritten. For this reason, our proof will not care about the precise order of updates and queries in the right interval, and there will be no difference between the real and simulated queries. Let the uniformly-spaced set of update indices be $S = \{i_1, i_2, \dots\}$. We let B_j be the set of queries from the right interval (either real or simulated) whose random column is in $[i_j, i_{j+1}-1]$. For notational convenience, we write $H(B_j)$ for the entropy of the correct parameters to the set of queries B_j. Basic information theory states that $H(\bigcup_j B_j \mid Z) = \sum_j H(B_j \mid B_1, \dots, B_{j-1}, Z)$. Thus, to prove our lower bound, it suffices to show $H(B_j \mid B_1, \dots, B_{j-1}, Z) = \Omega(\sqrt{n} \lg n)$ for all j.

Let Z_j be a random variable describing Z and, in addition, the random permutations for all updates in the left interval with indices below i_j. Also, if there are any updates with indices in $[i_j + 1, i_{j+1} - 1]$, include their permutation in Z_j (these are updates outside the uniformly-spaced set). Note that $H(B_j \mid B_1, \dots, B_{j-1}, Z) \geq H(B_j \mid Z_j)$, because conditioning on Z_j also fixes the correct parameters for queries in $B_1, \dots, B_{j-1}$.

Now let us look at a query from B_j. The query picks a random node in the first column. All permutations before column i_j are fixed through Z_j, so we can trace the path of the random node until it enters box i_j. Assume we have the correct parameter of the query, i.e., the node from column k to which the initial node is connected. Permutations between column i_j and i_{j+1} are also fixed by Z_j, so we can trace back this node until the exit of box i_j. Thus, knowing the correct parameter is equivalent to knowing some point values of the permutation i_j. As long as the nodes chosen in the first column are distinct, we will learn new point values. If we query d distinct point values of the random permutation, the entropy of the correct parameters is $\Omega(d \lg n)$, for any d.

Now imagine an experiment choosing the queries sequentially. This describes a random walk for d. In each step, d may remain constant or it may be incremented. Becuase of the uniform spacing, the probability that a query ends up in B_j is $\Omega(\frac{\sqrt{n}}{BL})$. If $d \leq \sqrt{n}/2$, with probability at least a half, the node chosen in the first column is new. Then, for $d \leq \sqrt{n}/2$, the probability that d is incremented in $\Omega(\frac{\sqrt{n}}{BL})$. We do BL independent random steps, and we are interested in the expected value of d at the end. The waiting time until d is incremented is $O(\frac{BL}{\sqrt{n}})$. For a sufficiently small constant ε, the expected time until d reaches $\varepsilon\sqrt{n}$ is $\frac{1}{2}BL$. Then, with probability at least a half, $d \geq \varepsilon\sqrt{n}$ after BL steps. This is implies the expected value of d after BL steps is $\Omega(\sqrt{n})$, so $H(B_j \mid Z_j) = \Omega(\sqrt{n} \lg n)$. □

To use this lemma, we construct a tree with a branching factor $\beta \geq 2B\frac{t_u}{t_q}$, rounded to the next power of two. The right interval is formed by a node, and the left interval by the node's left siblings. We only consider the case when the node is among the right half of its siblings. Now we argue there is a uniformly-spaced subset among the indices updated in the left interval. Note that these include all indices from the first half of siblings. Because β is a power of two, a root-to-leaf path in the tree is tracing a bit representation of the leaf's index, in chunks of $\log_2 \beta$ bits. Because update indices are the reverse of the leaf's index, all the leaves in the subtrees of the first half of the children have the same low order bits in the indices. On the other hand, the high order bits assume all possible values. So the indices from the first half of the children are always a uniformly-spaced subset of indices.

Now we can apply Lemma 7.2, and we sum over all nodes of the tree to obtain our lower bound. By the analysis in the previous section, the sum of the s terms only changes the bound by a constant factor. The $\Omega(L)$ term of the lower bound is linear in the number of queries, so summing over all levels we obtain $E[T] = \Omega(\sqrt{n} \cdot \frac{t_u}{t_q}\sqrt{n} \cdot \log_\beta n)$. Because $E[T] = \sqrt{n}\left(t_u\sqrt{n} + t_q\frac{t_u}{t_q}\sqrt{n}\right) = O(\sqrt{n} \cdot t_u\sqrt{n})$, we obtain $t_q = \Omega(\log_\beta n)$, which is our desired lower bound.

8. Upper Bounds for the Partial Sums Problem. As mentioned before, our partial-sums data structure can support a harder variant of updates. We will allow the $A[i]$'s to be arbitrary b-bit integers, while $\text{UPDATE}(i, \Delta)$ implements the operation $A[i] \leftarrow A[i] + \Delta$, where Δ is a δ-bit (signed) integer.

Our data structure is based on a balanced tree of branching factor B (to be determined) with the elements of the array $A[1\,..\,n]$ in the leaves. Assume we pick B such that we can support constant-time operations for the partial-sums problem in an array of size B. Then, we can hold an array of size B in every node, where each element is the total of the leaves in one of the B subtrees of our node. All three operations in the large data structure translate into a sequence of operations on the small data structures of the nodes along a root-to-leaf path. Thus, the running time is $O(\log_B n)$. We will show how to handle $B = \Theta(\min\{b/\delta, b^{1/5}\})$. Then $\lg B = \Theta(\lg(b/\delta))$, which implies our upper bound.

It remains to describe the basic building block, i.e., a constant-time solution for arrays of B elements. We now give a simple solution for UPDATE and SUM. In the next section, we develop the ideas necessary to support SELECT. We will conceptually maintain an array of partial sums $S[1\,..\,B]$, where $S[k] = \sum_{i=1}^{k} A[i]$. To make it possible to support UPDATE in constant time, we maintain the array as two separate components, $V[1\,..\,B]$ and $T[1\,..\,B]$, such that $S[i] = V[i] + T[i]$. The array V will hold values of S that were valid at some point in the past, while more recent updates are reflected only in T. We can use Dietz's incremental rebuilding scheme [Die89] to maintain every element of B relatively up-to-date: on the t^{th} UPDATE, we set $V[t \bmod B] \leftarrow V[t \bmod B] + T[t \bmod B]$ and $T[t \bmod B] \leftarrow 0$. This scheme guarantees that every element in T is affected by at most B updates, and thus is bounded in absolute value by $B \cdot 2^\delta$.

The key idea is to pack T in a machine word. We represent each $T[i]$ by a range of $O(\delta + \lg n)$ bits from the word, with one zero bit of padding between elements. Elements in T can also be negative; in this case, each value will be represented in the standard two's complement form on its corresponding range of bits. Packing T in a word is possible as long as $B = O\left(\frac{b}{\delta + \lg b}\right)$. We can read and write an element of T using a constant number of standard RAM operations (bitwise boolean operations and shift operations).

To complete our solution, we need to implement UPDATE in constant time. Using the packed representation, we can add a given value to all elements $V[i]$, $i \geq k$, in constant time. Refer to Figure 8.1. First, we create a word with the value to be added appearing in all positions corresponding to the elements of V that need to be changed. We can compute this word using a multiplication by an appropriate binary pattern. The result is then added to the packed representation of V; all the needed additions are performed in one step, using word-level parallelism. Because we are representing negative quantities in two's complement, additions may carry over, and

$V[4]$	0	$V[3]$	0	$V[2]$	0	$V[1]$	0	$V[0]$		old packed representation of V
00001	0	00001	0	00001	0	00001	0	00001		constant pattern
00001	0	00001	0	00001	0	00000	0	00000		shift right, then left by same amount
								Δ		argument given to UPDATE
Δ	0	Δ	0	Δ	0	00000	0	00000		multiply the last two values
$V'[4]$	?	$V'[3]$	?	$V'[2]$	?	$V[1]$	?	$V[0]$		add to the packed representation of V
11111	0	11111	0	11111	0	11111	0	11111		constant cleaning pattern
$V'[4]$	0	$V'[3]$	0	$V'[2]$	0	$V'[1]$	0	$V'[0]$		final value of V, obtained by bitwise AND

FIG. 8.1. *Performing* UPDATE$(2, \Delta)$ *at the word level. Here V has 5 elements, each 5 bits long.*

set the padding bits between elements; we therefore force these buffer bits to zero using a bitwise AND with an appropriate constant mask.

8.1. Selecting in Small Arrays. To support SELECT, we use the classic result of Fredman and Willard [FW93] that forms the basis of their fusion-tree data structure. Their result has the following black-box functionality: for $B = O(b^{1/5})$, we can construct a data structure that can answer successor queries on a static array of B integers in constant time. As demonstrated in [AMT99], the lookup tables used by the original data structure can be eliminated, if we perform a second query in the sketch representation of the array. The data structure can then be constructed in $O(B^4)$ time.

As before, we break partial sums into the arrays V and T. We store a fusion structure that can answer successor queries in V. Because the fusion structure is static, we abandon the incremental rebuilding of V, in favor of periodic global rebuilding. By the standard deamortization of global rebuilding [dBSvKO00], we can then obtain worst-case bounds. Our strategy is to rebuild the data structure completely every B^4 operations: we set $V[i] \leftarrow V[i] + T[i]$ and $T[i] \leftarrow 0$, for all i, and rebuild the fusion structure over V. While servicing a SELECT that doesn't occur immediately after a rebuild, the successor in V found by the fusion structure might not be the appropriate answer to the SELECT query, because of recent updates. We will describe shortly how the correct answer can be computed by also examining the array T; the key realization is that the real successor must be close to the successor in V in terms of their partial sums.

Central to our solution is the way we rebuild the data structure every n^4 operations. We begin by splitting S into runs of elements satisfying $S[i+1] - S[i] < B^4 \cdot 2^\delta$; recall that we must have $S[i] < S[i+1]$ for the SELECT problem. We denote by $\text{rep}(i)$ the first element of the run containing i (the representative of the run); also let $\text{len}(i)$ be the length of the run containing i. Each of these arrays can be packed in a word, because we already limited ourselves to $B = O(b^{1/5})$. Finally, we let every $V[i] \leftarrow V[\text{rep}(i)]$ and $T[i] \leftarrow S[i] - V[\text{rep}(i)]$. Conveniently, T can still be packed in a word. Indeed, the value stored in an element after a rebuild is at most $B \cdot \left(B^4 \cdot 2^\delta\right)$, and it can subsequently change by less than $B^4 \cdot 2^\delta$. Therefore, it takes $O(\lg B + \delta)$ bits to represent an element of C, so we only need to impose the condition that $B = O(\min\{b/\delta, b^{1/5}\})$.

It remains to show how SELECT(σ) can be answered. Let k denote the successor in V identified by the fusion structure; we have $V[k-1] < \sigma \leq V[k]$. We know that k is the representative of a run, because all elements of a run have equal values in V. By construction, runs are separated by gaps of at least $B^4 \cdot 2^\delta$, which cannot be closed by B^4 updates. Thus, the answer to the query must be either an index in the run starting at k, or an index in the run ending at $k-1$, or exactly equal to $k+\text{len}(k)$. We can distinguish between these cases in constant time, using two calls to SUM followed by comparisons. If we identify the correct answer as exactly $k+\text{len}(k)$, we are done.

Otherwise, assume by symmetry that the answer is an index in the run starting at k. Because elements of a run have equal values of V, our task is to identify the unique index i in the run satisfying $T[i-1] < \sigma - V[k] \leq T[i]$. Now we can employ word-level parallelism to compare all elements in T with $\sigma - V[k]$ in constant time. This is similar to a problem discussed by Fredman and Willard [FW93], but we must also handle negative quantities. The solution is to subtract $\sigma - V[k]$ in parallel from all elements in T; if elements of T are oversized by 1 bit, we can avoid overflow. The sign bits of every element then give the results of the comparisons. The answer to the query can be found by summing up the sign bits corresponding to elements in our run, which indicates how many elements in the run were smaller than $\sigma - V[k]$. Because these bits are separated by more than $\lg b$ zeroes, we can sum them up using a multiplication with a constant pattern, as described in [FW93].

9. Reductions To Other Dynamic Graph Problems. It is relatively easy to dress up dynamic connectivity as other dynamic graph problems, obtaining logarithmic lower bounds for these. Most problems on undirected graphs admit polylogarithmic solutions, so such lower bounds are interesting. The problems discussed in this section are only meant as examples, and not as an exhaustive list of possible reductions.

9.1. Connectivity of the Entire Graph. The problem is to maintain a dynamic graph along with the answer to the question "is the entire graph connected?". We obtain a lower bound of $\Omega(\lg n)$ even for plane graphs, which implies the same lower bound for counting connected components. The dynamic connectivity algorithms mentioned in the introduction can also maintain the number of connected components, so the same almost-tight upper bounds hold for this problem.

We use the same graph as in the dynamic connectivity lower bound, except that we add a new vertex s which is connected to all nodes from the first column. The updates in the connectivity problem translate to identical updates in our current problem. The hard instance of connectivity asks queries between a vertex u on the first column, and an arbitrary vertex v. To simulate these, we disconnect u from s, connect v to s, and ask whether the entire graph is connected; after this, we undo the two changes to the graph. If u and v were on distinct paths, u's path will now be disconnected from the rest of the graph. Otherwise, the edge (v, s) will reconnect the path to the rest of the graph.

The graph we consider is a tree, so it is plane regardless of the embedding of the vertices. During a query, if u and v are on the same path, we create an ephemeral cycle. However, the (v, s) edge can simply be routed along the old path $s \to u \rightsquigarrow v$, so the graph remains plane.

9.2. Dynamic MSF. The problem is to maintain the cost of the minimum spanning forest in a weighted dynamic graph. The problem can be solved in $O(\lg^4 n)$ time per operation [HdLT01]. In plane graphs, the problem admits a solution in time $O(\lg n)$ [EIT$^+$92]. We obtain a lower bound of $\Omega(\lg n)$, which holds even for plane graphs with unit weights. Our bound follows immediately from the previous bound. If all edges have unit weight and the graph is connected, the weight of the MSF is $n-1$. If the graph is disconnected, the weight of the MSF is strictly smaller.

9.3. Dynamic Planarity Testing. The problem is to maintain a dynamic plane graph, and test whether inserting a given edge would destroy planarity. Actual insertions always maintain planarity; an edge (u, v) is given along with an order inside the set of edges adjacent to u and v. The problem can be solved in $O(\lg^2 n)$ time per operation [IPH93]. A lower bound of $\Omega(\lg n/\lg\lg n)$ appears in [FH98]. We obtain a lower bound of $\Omega(\lg n)$.

Because the graph from our lower bound proof is always a collection of disjoint paths, it is plane under any embedding. Consider on the side the complete bipartite graph $K_{3,3}$, from which an edge (s, t) is removed. Without that edge, this annex graph is also planar. To implement connectivity queries between two nodes u and v, we first insert the edge (u, s) temporarily, and then query whether inserting the edge (v, t) would destroy planarity. If u and v are on distinct paths, the graph created by adding (u, s) and (v, t) is planar, and can be embedded for any relative order of these two edges (the edges of $K_{3,3} \setminus \{(s, t)\}$ can simply go around the two paths containing u and v). If u and v are on the same path, we would be creating a subdivision (graph expansion) of $K_{3,3}$, so the graph would no longer be planar (by Kuratowski's theorem).

10. Open Problems. This paper provides powerful techniques for understanding problems which have complexity around $\Theta(\lg n)$. The chronogram technique had already proven effective for problems with complexity $\Theta(\frac{\lg n}{\lg\lg n})$. However, current techniques seem helpless either below or above these thresholds. Below this regime, we have integer search problems, such as priority queues. Looking at higher complexities, we find many important problems which seem to have polylogarithmic complexity (such as range queries in higher dimensions) or even $n^{\Omega(1)}$ complexity (dynamic problems on directed graphs). It is also an important complexity theoretic challenge to obtain an $\omega(\lg n)$ lower bound for a dynamic language membership problem.

It is also worth noting that our bounds do not bring a complete understanding of the partial-sums problem when $\delta = o(b)$. First, we cannot prove a tight bound for VERIFY-SUM. A bound of $\Omega(\lg n/\lg b)$, for any δ, is implicit in [HR03], and can also be reproved using our technique. Second, we do not have a good understanding of the possible trade-offs. For SELECT, this seems a thorny issue, because of the interaction with the predecessor problem. Even for SUM, we do not know what bounds are possible in the range $t_u < t_q$. It is tempting to think that the right bound is $t_u(\lg \frac{t_q}{t_u} + \lg \frac{b}{\delta}) = \Theta(\lg n)$, by symmetry with the case $t_u > t_q$. However, buffer trees [Arg03] give better bounds for some choices of parameters, e.g. when $b = \Omega(\lg^{1+\varepsilon} n)$. This problem seems to touch on a fundamental issue: a good lower bound apparently needs to argue that the data structure has to recover a lot of information about the indices of updates, in addition to the Δ values.

It would be very interesting to obtain a logarithmic upper bound for dynamic connectivity, matching our lower bound. It would also be interesting to determine the complexity of decremental connectivity. For this problem, at least our trade-off lower bound cannot hold, because [HK99] gave a solution with polylogarithmic updates and constant query time.

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