Algorithmic Graph Minor Theory: Improved Grid Minor Bounds and Wagner's Contraction

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Abstract. We explore the three main avenues of research still unsolved in the algorithmic graph-minor theory literature, which all stem from a key min-max relation between the treewidth of a graph and its largest grid minor. This min-max relation is a keystone of the Graph Minor Theory of Robertson and Seymour, which ultimately proves Wagner's Conjecture about the structure of minor-closed graph properties.

First, we obtain the only known polynomial min-max relation for graphs that do not exclude any fixed minor, namely, map graphs and power graphs. Second, we obtain explicit (and improved) bounds on the min-max relation for an important class of graphs excluding a minor, namely, $K_{3,k}$-minor-free graphs, using new techniques that do not rely on Graph Minor Theory. These two avenues lead to faster fixed-parameter algorithms for two families of graph problems, called minor-bidimensional and contraction-bidimensional parameters. Third, we disprove a variation of Wagner's Conjecture for the case of graph contractions in general graphs, and in a sense characterize which graphs satisfy the variation. This result demonstrates the limitations of a general theory of algorithms for the family of contraction-closed problems (which includes, for example, the celebrated dominating-set problem). If this conjecture had been true, we would have had an extremely powerful tool for proving the existence of efficient algorithms for any contraction-closed problem, like we do for minor-closed problems via Graph Minor Theory.

1 Introduction

Graph Minor Theory is a seminal body of work in graph theory, developed by Robertson and Seymour in a series of over 20 papers spanning the last 20 years. The original goal of this work, now achieved, was to prove Wagner's Conjecture [39], which can be stated as follows: every minor-closed graph property (preserved under taking of minors) is characterized by a finite set of forbidden minors. This theorem has a powerful algorithmic consequence: every minor-closed graph property can be decided by a polynomial-time algorithm. A keystone in the proof of these theorems, and many other theorems, is a grid-minor theorem [37]: any graph of treewidth at least some $f(r)$ is guaranteed to have the $r \times r$ grid graph as a minor. Such grid-minor theorems have also played a key role for many algorithmic applications, in particular via the bidimensionality theory (e.g., [20,13,15,11,18,17,19]), including many approximation algorithms, PTASs, and fixed-parameter algorithms.

The grid-minor theorem of [37] has been extended, improved, and re-proved. The best bound known for general graphs is superexponential: every graph of treewidth more than 20^{2r^5} has an $r \times r$ grid minor [43]. This bound is usually not strong enough to derive efficient algorithms. Robertson et al. [43] conjecture that the bound on $f(r)$ can be improved to a polynomial $r^{\Theta(1)}$; the best known lower bound is $\Omega(r^2 \lg r)$. A tight linear upper bound was recently established for graphs excluding any fixed minor H: every H-minor-free graph of treewidth at least $c_H\, r$ has an $r \times r$ grid minor, for some constant c_H [18]. This bound leads to many powerful algorithmic results on H-minor-free graphs [18,17,19].

Three major problems remain in the literature with respect to these grid-minor theorems in particular, and algorithmic graph-minor theory in general. We address all three of these problems in this paper.

First, to what extent can we generalize algorithmic graph-minor results to graphs that do not exclude a fixed minor H? In particular, for what classes of graphs can the grid-minor theorem be improved from the general superexponential bound to a bound that would be useful for algorithms? To this end, we present polynomial grid-minor theorems for two classes of graphs that can have arbitrarily large cliques (and therefore exclude no fixed minors). One class, *map graphs*, is an important generalization of planar graphs introduced by Chen, Grigni, and Papadimitriou [10], characterized via a polynomial recognition algorithm by Thorup [45], and studied extensively in particular in the context of subexponential fixed-parameter algorithms and PTASs for specific domination problems [12,9]. The other class, *power graphs*, e.g., fixed powers of H-minor-free graphs (or even map graphs), have been well-studied since the time of the Floyd-Warshall algorithm.

Second, even for H-minor-free graphs, how large is the constant c_H in the grid-minor theorem? In particular, how does it depend on H? This constant is particularly important because it is in the exponent of the running times of many algorithms. The current results (e.g., [18]) heavily depend on Graph Minor Theory, most of which lacks explicit bounds and is believed to have very large bounds. (To quote David Johnson [32], "for any instance $G = (V, E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour's." He estimates one constant in an algorithm for testing for a fixed minor H to be roughly $2 \uparrow 2^{2^{2^{2 \uparrow (2 \uparrow \Theta(|V(H)|))}}}$, where $2 \uparrow n$ denotes a tower $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ involving n 2's.) For this reason, improving the constants, even for special classes of graphs, and presumably using different approaches from Graph Minors, is an important theoretical and practical challenge. To this end, we give explicit bounds for the case of $K_{3,k}$-minor-free graphs, an important class of apex-minor-free graphs (see, e.g., [4,7,26,27]). Our bounds are not too small but are a vast improvement over previous bounds (in particular, much smaller than $2 \uparrow |V(H)|$); in addition, the proof techniques are interesting in their own right, being disjoint from most of Graph Minors. To the best of our knowledge, this is the only grid-minor theorem with an explicit bound other than for planar graphs [43] and bounded-genus graphs [13]. Our theorem also leads to several algorithms with explicit and improved bounds on their running time.

Third, to what extent can we generalize algorithmic graph-minor results to graph contractions? Many graph optimization problems are closed (only decrease) under edge contractions, but not under edge deletions (i.e., minors). Examples include dominating set, traveling salesman, or even diameter. Bidimensionality theory has been extended to such contraction-closed problems for the case of apex-minor-free graphs; see, e.g., [11,13,18,17,23]. The basis for this work is a modified grid-minor theorem which states that any apex-minor-free graph of treewidth at least $f(r)$ can be contracted into an "augmented" $r \times r$ grid (e.g., allowing partial triangulation of the faces). The ultimate goal of this line of research, mentioned explicitly in [16,23], is to use this grid-contraction analog of the grid-minor theorem to develop a Graph Contraction Theory paralleling as much as possible of Graph Minor Theory. In particular, the most natural question is whether Wagner's Conjecture generalizes to contractions: is every contraction-closed graph property characterized by a finite set of excluded contractions? If this were true, it would generalize our algorithmic knowledge of minor-closed graph problems in a natural way to the vast array of contraction-closed graph problems. To this end, we unfortunately disprove this contraction version of Wagner's Conjecture, even for planar bounded-treewidth graphs. On the other hand, we prove that the conjecture holds for outerplanar graphs and triangulated planar graphs, which in some sense provides a tight characterization of graphs for which the conjecture holds.

Below we detail our results and techniques for each of these three problems.

1.1 Our Results and Techniques

Generalized grid-minor bounds. We establish polynomial relations between treewidth and grid minors for map graphs and for powers of graphs. We prove in Section 2 that any map graph of treewidth at least r^3 has an $\Omega(r) \times \Omega(r)$ grid minor. We prove in Section 3 that, for any graph class with a polynomial relation between treewidth and grid minors (such as H-minor-free graphs and map graphs), the family of kth powers of these graphs also has such a polynomial relation, where the polynomial degree is larger by just a constant, interestingly independent of k.

These results extend bidimensionality to map graphs and power graphs, improving the running times of a broad class of fixed-parameter algorithms for these graphs. See Section 4 for details on these algorithmic implications. Our results also build support for Robertson, Seymour, and Thomas's conjecture that all graphs have a polynomial relation between treewidth and grid minors [43].

Indeed, from our work, we refine the conjecture to state that all graphs of treewidth $\Omega(r^3)$ have an $\Omega(r) \times \Omega(r)$ grid minor, and that this bound is tight. The previous best treewidth-grid relations for map graphs and power graphs were given by the superexponential bound from [43].

The main technique behind these results is to use approximate min-max relations between treewidth and the size of a grid minor. In contrast, most previous work uses the seminal approximate min-max relation between treewidth and tangles or between branchwidth and tangles, proved by Robertson and Seymour [42]. We show that grids are powerful structures that are easy to work with. By bootstrapping, we use grids and their connections to treewidth even to prove relations between grids and treewidth.

Another example of the power of this technique is a result we obtain as a byproduct of our study of map graphs: every bounded-genus graph has treewidth within a constant factor of the treewidth of its dual. This is the first relation of this type for bounded-genus graphs. The result generalizes a conjecture of Seymour and Thomas [44] that, for planar graphs, the treewidth is within an additive 1 of the treewidth of the dual, which has apparently been proved in [35,5] using a complicated approach. Such a primal-dual treewidth relation is useful, e.g., for bounding the change in treewidth when performing operations in the dual. Our proof crucially uses the connections between treewidth and grid minors, and this approach leads to a relatively clean argument. The tools we use come from bidimensionality theory and graph contractions, even though the result is not explicitly about either.

Explicit (improved) grid-minor bounds. We prove in Section 5 that the constant c_H in the linear grid-minor bound for H-minor-free graphs can be bounded by an explicit function of $|V(H)|$ when $H = K_{3,k}$ for any k: for an explicit constant c, every $K_{3,k}$-minor-free graph of treewidth at least $c^k r$ has an $r \times r$ grid minor. This bound makes explicit and substantially improves the constants in the exponents of the running time of many fixed-parameter algorithms from bidimensionality theory [13,11,18] for such graphs. $K_{3,k}$-minor-free graphs play an important role as part of the family of apex-minor-free graphs that is disjoint from the family of single-crossing-minor-free graphs (for which there exist a powerful decomposition in terms of planar graphs and bounded-treewidth graphs [41,20]). Here the *family of $\mathcal{X}$-minor-free graphs* denotes the set of X-minor-free graphs for any fixed graph X in the class $\mathcal{X}$. $K_{3,k}$ is an *apex graph* in the sense that it has a vertex whose removal leaves a planar graph. For $k \geq 7$, $K_{3,k}$ is not a *single-crossing graph* in the sense of being a minor of a graph that can be drawn in the plane with at most one crossing: $K_{3,k}$ has genus at least $(k-2)/4$, but a single-crossing graph has genus at most 1 (because genus is closed under minors).

There are several structural theorems concerning $K_{3,k}$-minor-free graphs. According to Robertson and Seymour (personal communication—see [7]), $K_{3,k}$-minor-free graphs were the first step toward their core result of decomposing graphs excluding a fixed minor into graphs almost-embeddable into bounded-genus surfaces, because $K_{3,k}$-minor-free graphs can have arbitrarily large genus. Oporowski, Oxley, and Thomas [36] proved that any large 3-connected $K_{3,k}$-

minor-free graph has a large wheel as a minor. Böhme, Kawarabayashi, Maharry, and Mohar [3] proved that any large 7-connected graph has a $K_{3,k}$ minor, and that the connectivity 7 is best possible. Eppstein [26,27] proved that a subgraph P has a linear bound on the number of times it can occur in $K_{3,k}$-minor-free graphs if and only if P is 3-connected.

Our explicit linear grid-minor bound is based on an approach of Diestel et al. [24] combined with arguments in [4,3] to find a $K_{3,k}$ minor. Using similar techniques we also give explicit bounds on treewidth for a theorem decomposing a single-crossing-minor-free graph into planar graphs and bounded-treewidth graphs [41,20], when the single-crossing graph is $K_{3,4}$ or K_6^- (K_6 minus one edge). Both proofs must avoid Graph Minor Theory to obtain the first explicit bounds of their kind.

Contraction version of Wagner's Conjecture. Wagner's Conjecture, proved in [39], is a powerful and very general tool for establishing the existence of polynomial-time algorithms; see, e.g., [28]. Combining this theorem with the $O(n^3)$-time algorithm for testing whether a graph has a fixed minor H [38], every minor-closed property has an $O(n^3)$-time decision algorithm which tests for the finite set of excluded minors. Although these results are existential, because the finite set of excluded minors is not known for many minor-closed properties, polynomial-time algorithms can often be constructed [14].

A natural goal is to try to generalize these results even further, to handle all contraction-closed properties, which include the decision versions of many important graph optimization problems such as dominating set and traveling salesman, as well as combinatorial properties such as diameter. Unfortunately, we show in Section 6 that the contraction version of Wagner's Conjecture is not true: there is a contraction-closed property that has no complete finite set of excluded contractions. Our counterexample has an infinite set of excluded contractions all of which are planar bounded-treewidth graphs. On the other hand, we show that the contraction version of Wagner's Conjecture holds for trees, triangulated planar graphs, and 2-connected outerplanar graphs: any contraction-closed property characterized by an infinite set of such graphs as contractions can be characterized by a finite set of such graphs as contractions. Thus we nearly characterize the set of graphs for which the contraction version of Wagner Conjecture's is true. The proof for outerplanar graphs is the most complicated, and uses Higman's theorem on well-quasi-ordering [31].

The reader is referred to the full version of this paper (available from the first author's website) for the proofs. See also [16] for relevant definitions.

2 Treewidth-Grid Relation for Map Graphs

In this section we prove a polynomial relation between the treewidth of a map graph and the size of the largest grid minor. The main idea is to relate the treewidth of the map graph, the treewidth of the radial graph, the treewidth of the dual graph, and the treewidth of the union graph.

Theorem 1. *If the treewidth of the map graph M is r^3, then it has an $\Omega(r) \times \Omega(r)$ grid as a minor.*

This theorem cannot be improved from $\Omega(r^3)$ to anything $o(r^2)$:

Proposition 1. *There are map graphs whose treewidth is $r^2 - 1$ and whose largest grid minor is $r \times r$.*

Robertson, Seymour, and Thomas [43] prove a stronger lower bound of $\Theta(r^2 \lg r)$ but only for the case of general graphs.

3 Treewidth-Grid Relation for Power Graphs

In this section we prove a polynomial relation between the treewidth of a power graph and the size of the largest grid minor. The technique here is quite different, analyzing how a radius-r neighborhood in the graph can be covered by radius-$(r/2)$ neighborhoods—a kind of "sphere packing" argument.

Theorem 2. *Suppose that, if graph G has treewidth at least cr^α for constants $c, \alpha > 0$, then G has an $r \times r$ grid minor. For any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least $cr^{\alpha+4}$ (respectively, $cr^{\alpha+6}$), then it has an $r \times r$ grid minor.*

We have the following immediate consequence of Theorems 1 and 2 and the grid-minor theorem of [18] mentioned in the introduction:

Corollary 1. *For any H-minor-free graph G, and for any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least r^5 (respectively, r^7), then it has an $\Omega(r) \times \Omega(r)$ grid minor. For any map graph G, and for any even (respectively, odd) integer $k \geq 1$, if G^k has treewidth at least r^7 (respectively, r^9), then it has an $\Omega(r) \times \Omega(r)$ grid minor.*

4 Treewidth-Grid Relations: Algorithmic and Combinatorial Applications

Our treewidth-grid relations have several useful consequences with respect to fixed-parameter algorithms, minor-bidimensionality, and parameter-treewidth bounds.

A *fixed-parameter algorithm* is an algorithm for computing a parameter $P(G)$ of a graph G whose running time is $h(P(G))\, n^{O(1)}$ for some function h. A typical function h for many fixed-parameter algorithms is $h(k) = 2^{O(k)}$. A celebrated example of a fixed-parameter-tractable problem is vertex cover, asking whether an input graph has at most k vertices that are incident to all its edges, which admits a solution as fast as $O(kn + 1.285^k)$ [8]. For more results about fixed-parameter tractability and intractability, see the book of Downey and Fellows [25].

A major recent approach for obtaining efficient fixed-parameter algorithms is through "parameter-treewidth bounds", a notion at the heart of bidimensionality. A *parameter-treewidth bound* is an upper bound $f(k)$ on the treewidth of a graph with parameter value k. Typically, $f(k)$ is polynomial in k. Parameter-treewidth bounds have been established for many parameters; see, e.g., [1,33,29,2,6,34,30,12,20,21,22,11,15,13]. Essentially all of these bounds can be obtained from the general theory of bidimensional parameters (see, e.g., [16]). Thus bidimensionality is the most powerful method so far for establishing parameter-treewidth bounds, encompassing all such previous results for H-minor-free graphs. However, all of these results are limited to graphs that exclude a fixed minor.

A parameter is *minor-bidimensional* if it is at least $g(r)$ in the $r \times r$ grid graph and if the parameter does not increase when taking minors. Examples of minor-bidimensional parameters include the number of vertices and the size of various structures, e.g., feedback vertex set, vertex cover, minimum maximal matching, face cover, and a series of vertex-removal parameters. Tight parameter-treewidth bounds have been established for all minor-bidimensional parameters in H-minor-free graphs for any fixed graph H [18,11,13].

Our results provide polynomial parameter-treewidth bounds for all minor-bidimensional parameters in map graphs and power graphs:

Theorem 3. *For any minor-bidimensional parameter P which is at least $g(r)$ in the $r \times r$ grid, every map graph G has treewidth $\mathrm{tw}(G) = O(g^{-1}(P(G)))^3$. More generally suppose that, if graph G has treewidth at least cr^α for constants $c, \alpha > 0$, then G has an $r \times r$ grid minor. Then, for any even (respectively, odd) integer $k \geq 1$, G^k has treewidth $\mathrm{tw}(G) = O(g^{-1}(P(G)))^{\alpha+4}$ (respectively, $\mathrm{tw}(G) = O(g^{-1}(P(G)))^{\alpha+6}$). In particular, for H-minor-free graphs G, and for any even (respectively, odd) integer $k \geq 1$, G^k has treewidth $\mathrm{tw}(G) = O(g^{-1}(P(G)))^5$ (respectively, $\mathrm{tw}(G) = O(g^{-1}(P(G)))^7$).*

This result naturally leads to a collection of fixed-parameter algorithms, using commonly available algorithms for graphs of bounded treewidth:

Corollary 2. *Consider a parameter P that can be computed on a graph G in $h(w)\, n^{O(1)}$ time given a tree decomposition of G of width at most w. If P is minor-bidimensional and at least $g(r)$ in the $r \times r$ grid, then there is an algorithm computing P on any map graph or power graph G with running time $[h(O(g^{-1}(k))^\beta) + 2^{O(g^{-1}(k))^\beta}]\, n^{O(1)}$, where β is the degree of $O(g^{-1}(P(G))$ in the polynomial treewidth bound from Theorem 3. In particular, if $h(w) = 2^{O(w)}$ and $g(k) = \Omega(k^2)$, then the running time is $2^{O(k^{\beta/2})} n^{O(1)}$.*

The proofs of these consequences follow directly from combining [11] with Theorems 1 and 2 below.

In contrast, the best previous results for this general family of problems in these graph families have running times $[h(2^{O(g^{-1}(k))^5}) + 2^{2^{O(g^{-1}(k))^5}}]\, n^{O(1)}$ [11,14].

5 Improved Grid Minor Bounds for $K_{3,k}$

Recall that every graph excluding a fixed minor H having treewidth at least $c_H r$ has the $r \times r$ grid as a minor [18]. The main result of this section is an explicit bound on c_H when $H = K_{3,k}$ for any k:

Theorem 4. *Suppose G is a graph with no $K_{3,k}$-minor. If the treewidth is at least $20^{4k}r$, then G has an $r \times r$ grid minor.*

In [43], it was shown that if the treewidth is at least $f(r) \geq 20^{2^r}$, then G has an $r \times r$ grid as a minor. Our second theorems use this result to show the following. A *separation* of G is an ordered pair (A, B) of subgraphs of G such that $A \cup B = G$ and there are no edges between $A - B$ and $B - A$. Its *order* is $|A \cap B|$. Suppose G has a separation (A, B) of order k. Let A^+ be the graph obtained from A by adding edges joining every pair of vertices in $V(A) \cap V(B)$. Let B^+ be obtained from B similarly. We say that G is the *k-sum* of A^+ and B^+. If both A^+ and B^+ are minors of G other than G itself, we say that G is the *proper k-sum* of A^+ and B^+.

Using similar techniques as the theorem above, we prove the following two structural results decomposing $K_{3,4}$-minor-free and K_6^--minor-free graphs into proper k-sums:

Theorem 5. *Every $K_{3,4}$-minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most* $20^{2^{15}}$.

Theorem 6. *Every K_6^--minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most* $20^{2^{15}}$.

These theorems are explicit versions of the following decomposition result for general single-crossing-minor-free graphs (including $K_{3,4}$-minor-free and K_6^--minor-free graphs):

Theorem 7. [41] *For any fixed single-crossing graph H, there is a constant w_H such that every H-minor-free graph can be obtained via proper* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and graphs of treewidth at most w_H.*

This result heavily depends on Graph Minor Theory, so the treewidth bound w_H is huge—in fact, no explicit bound is known. Theorems 5 and 6 give reasonable bounds for the two instances of H we consider. Our proof of Theorem 5 uses a 15×15 grid minor together with the result in [40]. The latter result says roughly that, if there is a planar subgraph H in a non-planar graph G, then H has either a non-planar "jump" or "cross" in G such that the resulting graph is a minor of G. Our approach is to find a $K_{3,4}$-minor in a 13×13 grid minor plus some non-planar jump or cross. Similar techniques allow us to prove almost the same result for K_6^--free graphs in Theorem 6.

6 Contraction Version of Wagner's Conjecture

Motivated in particular by Kuratowski's Theorem characterizing planar graphs as graphs excluding $K_{3,3}$ and K_5 as minors, Wagner conjectured and Robertson and Seymour proved the following three results:

Theorem 8 (Wagner's Conjecture). [39] *For any infinite sequence $G_0, G_1, G_2, \ldots$ of graphs, there is a pair (i, j) such that $i < j$ and G_i is a minor of G_j.*

Corollary 3. [39] *Any minor-closed graph property*[4] *is characterized by a finite set of excluded minors.*

Corollary 4. [39,38] *Every minor-closed graph property can be decided in polynomial time.*

The important question we consider is whether these theorems hold when the notion of "minor" is replaced by "contraction". The motivation for this variation is that many graph properties are closed under contractions but not under minors (i.e., deletions). Examples include the decision problems associated with dominating set, edge dominating set, connected dominating set, diameter, etc.

One positive result along these lines is about minor-closed properties:

Theorem 9. *Any minor-closed graph property is characterized by a finite set of excluded contractions.*

For example, we obtain the following contraction version of Kuratowski's Theorem, using the construction of the previous theorem and observing that all other induced supergraphs of $K_{3,3}$ have K_5 as a contraction.

Corollary 5. *Planar graphs are characterized by a finite set of excluded contractions.*

Another positive result is that Wagner's Conjecture extends to contractions in the special case of trees. This result follows from the normal Wagner's Conjecture because a tree T_1 is a minor of another tree T_2 if and only if T_1 is a contraction of T_2:

Proposition 2. *For any infinite sequence $G_0, G_1, G_2, \ldots$ of trees, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Unfortunately, the contraction version of Wagner's Conjecture does not hold for general graphs:

Theorem 10. *There is an infinite sequence $G_0, G_1, G_2, \ldots$ of graphs such that, for every pair (i, j), $i \neq j$, G_i is not a contraction of G_j.*

[4] A *property* is simply a set of graphs, representing the graphs having the property.

Corollary 6. *There is a contraction-closed graph property that cannot be characterized by a finite set of excluded contractions.*

The graphs $G_i = K_{2,i+2}$ that form the counterexample of Theorem 10 and Corollary 6 are in some sense tight. Each G_i is a planar graph with faces of degree 4. If all faces are smaller, the contraction version of Wagner's Conjecture holds. A planar graph is *triangulated* if some planar embedding (or equivalently, every planar embedding) is triangulated, i.e., all faces have degree 3. Recall that the triangulated planar graphs are the maximal planar graphs, i.e., planar graphs in which no edges can be added while preserving planarity.

Theorem 11. *For any infinite sequence $G_0, G_1, G_2, \ldots$ of triangulated planar graphs, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Another sense in which the counterexample graphs $G_i = K_{2,i+2}$ are tight is that they are 2-connected and are 2-outerplanar, i.e., removing the (four) vertices on the outside face leaves an outerplanar graph (with all vertices on the new outside face). However, the contraction version of Wagner's Conjecture holds for 2-connected (1-)outerplanar graphs:

Theorem 12. *For any infinite sequence $G_0, G_1, G_2, \ldots$ of 2-connected embedded outerplanar graphs, there is a pair (i, j) such that $i < j$ and G_i is a contraction of G_j.*

Corollary 7. *Every contraction-closed graph property of trees, triangulated planar graphs, and/or 2-connected outerplanar graphs is characterized by a finite set of excluded contractions.*

We can use this result to prove the existence of a polynomial-time algorithm to decide any fixed contraction-closed property for trees and 2-connected outerplanar graphs, using a dynamic program that tests for a fixed graph contraction in a bounded-treewidth graph.

7 Open Problems and Conjectures

One of the main open problems is to close the gap between the best current upper and lower bounds relating treewidth and grid minors. For map graphs, it would be interesting to determine whether our analysis is tight, in particular, whether we can construct an example for which the $O(r^3)$ bound is tight. Such a construction would be very interesting because it would improve the best previous lower bound of $\Omega(r^2 \lg r)$ for general graphs [43]. We make the following stronger claim about general graphs:

Conjecture 1. For some constant $c > 0$, every graph with treewidth at least cr^3 has an $r \times r$ grid minor. Furthermore, this bound is tight: some graphs have treewidth $\Omega(r^3)$ and no $r \times r$ grid minor.

This conjecture is consistent with the belief of Robertson, Seymour, and Thomas [43] that the treewidth of general graphs is polynomial in the size of the largest grid minor.

We also conjecture that the contraction version of Wagner's Conjecture holds for k-outerplanar graphs for any fixed k. If this is true, it is particularly interesting that the property holds for k-outerplanar graphs, which have bounded treewidth, but does not work in general for bounded-treewidth graphs (as we have shown in Theorem 10).

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