Extended Abstract
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Unfolding Orthotubes with a Dual Hamiltonian Path

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Abstract

An *orthotube* consists of orthogonal boxes (e.g., unit cubes) glued face-to-face to form a path. In 1998, Biedl et al. showed that every orthotube has a *grid unfolding*: cutting along edges of the boxes so that the surface unfolds into a connected planar shape without overlap. We give a new algorithmic grid unfolding of orthotubes with the additional property that the rectangular faces are attached in a single path — a Hamiltonian path on the rectangular faces of the orthotube surface.

Keywords: unfolding polyhedra, orthotubes, Hamiltonicity, algorithm.

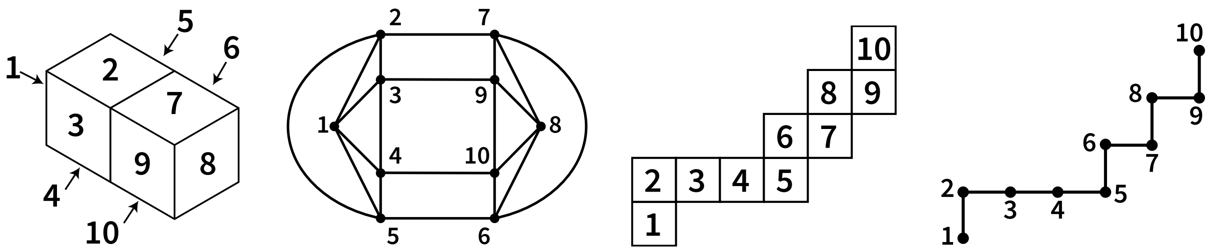
2010 MSC: Primary 68Q25; Secondary 52C99.

1 Introduction

Does every orthogonal polyhedron have a *grid unfolding*, that is, a cutting along edges of the induced grid (extending a plane through every face of the polyhedron) such that the remaining surface unfolds into a connected planar shape without overlap? This question remains unsolved over 20 years after this type of unfolding was introduced in 1998 [1]; see [9] for a survey and [3–5] for recent progress. This problem is in some sense the orthogonal nonconvex version of the older and more famous open problem of whether every convex polyhedron has an edge unfolding (cutting only along edges of the polyhedron) [6].

The first class of orthogonal polyhedra shown to have a grid unfolding is *orthotubes* [1], formed by gluing together a sequence of orthogonal boxes where every pair of consecutive boxes in the sequence share one face (and no other boxes share faces). Roughly speaking, this unfolding consists of a monotone dual-path of rectangular faces, with $O(1)$ rectangles attached above and below the path.

In this paper, we show that orthotubes have a grid unfolding with a stronger property we call *dual-Hamiltonicity*, where the unfolded shape consists of a single dual path of rectangular faces, as shown in Figure 1. More precisely, define the *face adjacency graph* to have a node for each rectangular face of a box, and connect two nodes by an edge whenever the corresponding rectangular faces share an edge. Then the unfolding is given by keeping attached the duals of edges that form a Hamiltonian path in the face adjacency graph. Implicitly, we take advantage of the fact that 4-connected planar graphs (which includes face adjacency graphs) have Hamiltonian cycles [2, 10].



(a) orthotube (b) face adjacency graph (c) dual-Hamiltonian unfolding (d) dual Hamiltonian path

Figure 1: An example of an orthotube, its face adjacency graph, dual-Hamiltonian unfolding corresponding to chain code $RSSLRLRL$, and the corresponding dual Hamiltonian path.

2 Unfolding

Our main result is the following:

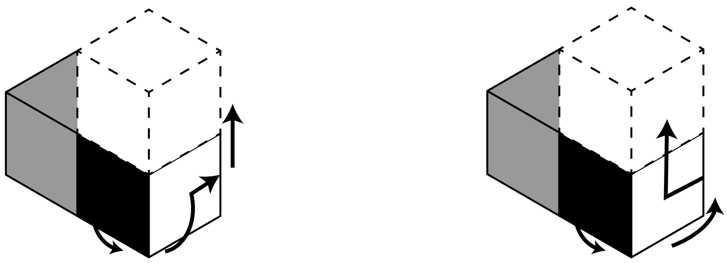
Theorem 2.1. *Given an orthotube, there is a Hamiltonian path of its face adjacency graph such that, if we follow the path, we get a non-overlapping unfolding.*

To prove the result, we use a "chain code" (similar to [7,8]) to represent a path on the surface of an orthotube. A *chain code* is an ordered sequence of the form $a_1 a_2 \ldots a_n$, where $a_i \in \{L, R, S\}$ represents an (intrinsic) left turn, right turn, or continuing straight to move from the ith face to the $(i+1)$st face.

For each chain code $c = a_1 a_2 \ldots a_n$, we define the cumulative quarter turning $\text{qturn}(c)$ to be $\sum_{i=1}^{n} \text{qturn}(a_i)$ where $\text{qturn}(R)$, $\text{qturn}(L)$, and $\text{qturn}(S)$ are $+1$, -1, and 0 respectively. To guarantee that our unfolding does not overlap, we prove the invariant that the qturn of any prefix of the chain code is in $\{-1, 0, 1\}$, that is, the unfolding proceeds monotonically in one direction.

2.1 Algorithm

At a high level, the algorithm creates a chain code for unfolding an orthotube up to three boxes at a time, to maintain the stronger condition that the unfolding so far has qturn $= 0$. We divide into cases based on each box's relative position to the next one, two, and sometimes three or four boxes (if they exist). Figure 2 shows two cases.



(a) $RLRL$ (b) $RSLR$

Figure 2: Two possible chain codes, $RLRL$ and $RSLR$, to unfold the current box from the black face, when the next box (white) is on top. The first code preserves qturn, so we use it unless the next next box (not drawn) is attached on the face that we try to enter on the next box (white). The second code adds 1 to qturn, and we show how in all cases to add an additional box to restore qturn to 0.

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