Algorithmic Transitions between Parallel Pleats

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Abstract: *We present a universal algorithm for constructing a locally flat-foldable crease pattern transitioning between two arbitrary sets of parallel pleats across a diagonal ridge crease. In other words, we generalize uniaxial ridge level shifters. We prove that such a transition is possible if and only if the number of input creases has the same parity on both sides, and the alternating sum of the intercepts of the input creases is the same on both sides. Finally, we show how such transition units can be useful for terminating dense bouncing.*

1 Introduction

In modern representational origami design, one widely used technique is a ***level shifter***, a structure that allows a crease pattern to transition from one axial height to another [Lang 11]. Level shifters are often used to create figures that are not a single uniform width — for example, insects with both thin legs and a wide shell, or human figures with both detailed hands and feet but also wide clothing. One of the most common and convenient ways to make a level shifter is by transitioning one set of parallel pleats to another set across a ridge, where the spacing of pleats might differ on one side and the other. This is known as a ***ridge level shifter***, and it is well known that one can easily shift from axial +1 to axial +2 using this method, as well as other integer heights [Abrashi 21].

However, beyond these well-known structures, existing level shifter methods do not have a way to generalize to a more complex spacing of pleats. In this paper, we will generalize ridge level shifters a step further by characterizing when parallel (axial) pleats of arbitrary spacing have a (locally) flat-foldable transition, and when they do, give an efficient algorithm to construct it. We also show an application of these generalized ridge level shifters to solving the problem of dense bouncing in axial origami design.

1.1 Problem and Result

We set up our problem as follows; refer to Figure 1. Let A and B each be a set of parallel semi-infinite creases (***rays***) starting on the x axis and forming an angle $0 < \theta < 90^\circ$ with the x axis, with A above the axis and B below the axis. Let $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_n\}$ be the x-intercepts of the creases in A and B respectively, sorted in strictly increasing order (from left to right). We assume the first crease of A and the first crease of B have the same mountain-valley direction, and the creases within each set alternate mountain/valley. We allow each crease in $A \cup B$ to be shortened, by starting the ray later than the x axis.

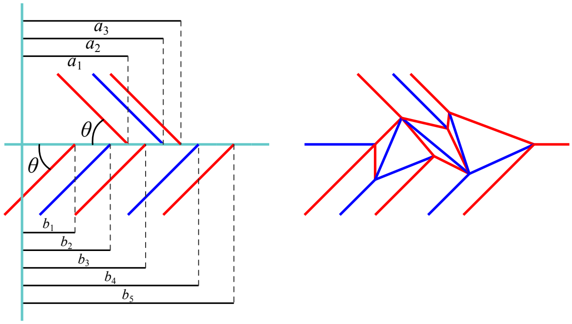

Figure 1: *The problem statement: given two input sets of parallel creases (left), can we construct a flat-foldable crease pattern that merges the two sets (right)? In this paper, we prove that this is possible if and only if* $a_1 - a_2 + a_3 - a_4 + \cdots a_m = b_1 - b_2 + b_3 - b_4 + \cdots b_n$ *and* $m \equiv n \pmod 2$.

We define the ***level-transition problem*** as follows: given inputs A, B, θ, when can we construct a flat-foldable crease pattern consisting of finitely many creases that includes an infinite subray of every crease in $A \cup B$, and leftward and rightward infinite subrays of the x axis? Thus we effectively combine A and B over a horizontal ridge.

Our main result is that a locally flat-foldable transition exists if and only if

$$a_1 - a_2 + a_3 - a_4 + \cdots a_m = b_1 - b_2 + b_3 - b_4 + \cdots b_n, \tag{1}$$

$$\text{and} \quad m \equiv n \pmod 2. \tag{2}$$

We call this the ***alternating-sum condition***. The parity equation (2) is also implied by assuming that $a_1, b_1 > 0$ (and thus all $a_i, b_i > 0$), because then the sign of $a_1 - a_2 + a_3 - \cdots a_m$ is positive for m odd and negative for m even.

2 Necessity

Theorem 1. *If a flat-foldable transition exists between two sets of parallel creases A and B with angle* θ*, then the alternating-sum condition is necessarily satisfied.*

Proof. We use the generalized Kawasaki's Theorem [Hull 20, Theorem 6.4] to prove this. Kawasaki's Theorem states that the alternating sum of angles around an interior vertex must be 0 (implying that every interior vertex must have an even number of creases). The generalized Kawasaki's Theorem extends from a single vertex to any closed curve C that does not intersect any vertices and stays interior to the paper; it states that, for any point, successively reflecting the point over each crease visited by C will bring the point back to where it started.

As shown in Figure 2, let C be a closed curve that intersects just the semi-infinite creases of a hypothetical flat-foldable transition: the (possibly trimmed) semi-infinite creases in A and B and the semi-infinite x-axis ridge creases. For convenience, we also require C to pass through the origin p, and assume all x-intercepts are positive.

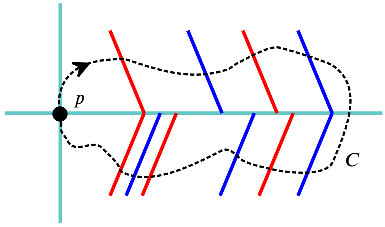

Figure 2: *A closed curve C that encloses the area to be filled by the transition, and arbitrary starting point p located at the origin.*

The generalized Kawasaki's Theorem implies that C intersects an even number of creases (in order to preserve the orientation of the plane). In our situation, C crosses $m+n+2$ creases where $m = |A|$ and $n = |B|$. Therefore, m and n must be either both odd or both even. This proves the parity equation (2) of the alternating-sum condition.

The x coordinate x_i of p after reflecting over the first i creases in A can be expressed as $x_i = (2a_1 - 2a_2 + 2a_3 - \cdots 2a_i)\sin\theta$. For C to ultimately return point p back to the origin, reflecting p over the creases in B from left to right must bring p to the same x coordinate x_m obtained by reflecting p over the creases in A from left to right. Putting together two similar computations, we must have

$$(2a_1 - 2a_2 + 2a_3 - \cdots 2a_m)\sin\theta = (2b_1 - 2b_2 + 2b_3 - \cdots 2b_n)\sin\theta.$$

Because $0 < \theta < 90^\circ$ and therefore $\sin\theta \neq 0$, we can divide $2\sin\theta$ from both sides. The equation then simplifies to equation (1) of the alternating-sum condition. □

As an aside, we can also prove that a flat-foldable transition cannot exist if the creases in A and B form different angles θ_A and θ_B (respectively) with the x axis. The generalized Kawasaki's Theorem around path C implies that reflecting p over creases in A and B not only should bring p to the same x coordinate, but also to opposite y coordinates, so that reflecting over the rightward x-axis crease causes the points to match. After reflecting p over the creases in A, we obtain the point

$$\big((2a_1 - 2a_2 + 2a_3 - \cdots 2a_m)\sin\theta_A,\ (2a_1 - 2a_2 + 2a_3 - \cdots 2a_m)\cos\theta_A\big),$$

and after reflecting p over the creases in B, we obtain the point

$$\big((2b_1 - 2b_2 + 2b_3 - \cdots 2b_n)\sin\theta_B,\ -(2b_1 - 2b_2 + 2b_3 - \cdots 2b_n)\cos\theta_B\big).$$

For these two points to have equal x coordinates and opposite y coordinates, they must have equal ratios of x coordinate over absolute y coordinate. In these ratios, the alternating sums of a_is or b_is cancel, and we are left with $\tan\theta_A = \tan\theta_B$, so $\theta_A = \theta_B$.

3 Sufficiency

We now prove inductively that, if the input creases satisfy the alternating-sum condition, then we can always construct a locally flat-foldable transition. In general, solutions are not unique. Here we will prove local flat foldability without considering mountain-valley assignments or self-intersection, which is equivalent to Kawasaki's Theorem [Demaine and O'Rourke 07, Hull 20].

At a high level, our transition construction has two mostly independent parts. First, we construct a graph of connections between the endpoints of the creases in A and B. Second, we construct the actual crease pattern by adjusting the locations of these endpoints (limited to "sliding" the vertex along its input crease) to make each vertex flat foldable.

3.1 1-to-*n* Construction

We start with the special case where A contains only one crease, which we call the ***1-to-n*** case; refer to Figure 3. By the parity equation (2), n must be odd. In this section, we derive the crease pattern's geometry using a sequence of equations.

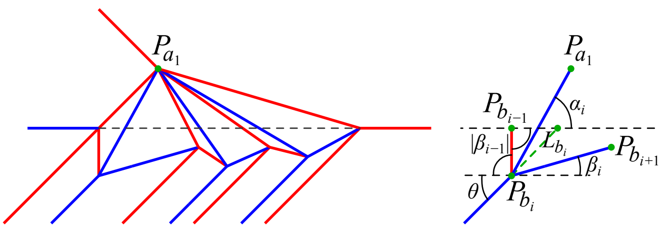

Figure 3: *The case where set A has only one crease. For a given point P_{b_i}, there is a crease connecting it to P_{a_1} that makes an angle α_i, and a crease connecting it to $P_{b_{i+1}}$ that makes an angle β_i. In this example, the segment from $P_{b_{i-1}}$ to P_{b_i} goes downward, so β_{i-1} is negative.*

Let P_{a_1} denote the endpoint of the a_1 semi-infinite crease (to be chosen), let P_{b_i} denote the endpoint of the b_i semi-infinite crease, and let L_{b_i} be the distance from that point to $(b_i, 0)$. Let $\alpha_i \in (-180^\circ, 180^\circ]$ denote the angle of the crease that connects P_{b_i} to P_{a_1}, and let $\beta_i \in (-180^\circ, 180^\circ]$ denote the angle of the crease that connects P_{b_i} to $P_{b_{i+1}}$, where both angles are measured relative to a rightward horizontal line.

Let us now develop constitutive relationships to relate the quantities β_i, L_{b_i}, and α_i. In general, each vertex P_{b_i} will be connected to four creases: the crease connecting P_{b_i} to P_{a_1} at an angle α_i, the crease going forwards from P_{b_i} to $P_{b_{i+1}}$ at an angle β_i, the crease going backwards from P_{b_i} to $P_{b_{i-1}}$ at an angle β_{i-1}, and its original input crease at an angle $180^\circ + \theta$ (measured relative to the positive x axis). Applying Kawasaki Theorem to this vertex P_{b_i}, as shown in Figure 3, gives us the following equation:

$$(\alpha_i - \beta_i) + (\theta - \beta_{i-1}) = 180^\circ, \qquad \text{i.e.,} \quad \beta_i = -\beta_{i-1} + \alpha_i - 180^\circ + \theta. \tag{3}$$

Vertex P_{b_i} is flat foldable if and only if equation (3) holds. This equation determines all β_is in terms of the first β_1, which is another free input parameter in the range $[-180^\circ + \theta, 0)$.

Next let us look at the flat foldability of vertex P_{a_1}, as shown in Figure 4. By

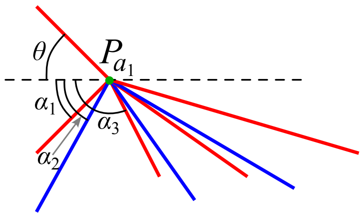

Figure 4: *A closeup of the vertex P_{a_1} and the angles of its creases. Kawasaki's Theorem requires that the alternating sum of the angles of the creases around P_{a_1} add up to 180°, giving us equation (4).*

Kawasaki's Theorem, P_{a_1} is flat foldable if and only if the following equation holds:

$$180^\circ = \theta + \alpha_1 + (\alpha_3 - \alpha_2) + \cdots + (\alpha_n - \alpha_{n-1})$$
$$\text{i.e.,}\quad \alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{n-1} + \alpha_n = 180^\circ - \theta. \tag{4}$$

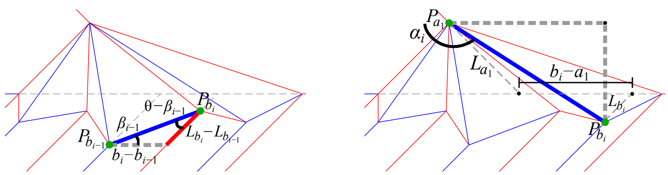

Figure 5: *Left: the length L_{b_i} for $i > 0$ can be derived by forming a triangle with the previous crease and an auxiliary horizontal line, then using the Law of Sines to yield equation (5). Right: the angle α_i can be derived by taking the difference in x and y coordinates of P_{b_i} and P_{a_1}, then taking the inverse tangent to yield equation (6).*

Next we geometrically derive expressions for L_{b_i} and α_i; refer to Figure 5. For any vertex P_{b_i}, we can form a triangle with vertex $P_{b_{i-1}}$ and a horizontal line through $P_{b_{i-1}}$, and use the Law of Sines to find that the length L_{b_i} is given by

$$\frac{L_{b_i} - L_{b_{i-1}}}{\sin\beta_{i-1}} = \frac{b_i - b_{i-1}}{\sin(\theta - \beta_{i-1})}, \qquad \text{i.e.,}\quad L_{b_i} = L_{b_{i-1}} - \frac{\sin\beta_{i-1}}{\sin(\theta - \beta_{i-1})}(b_i - b_{i-1}). \tag{5}$$

We define L_{b_1} to be 0 so that it meets the leftward infinite subray of the x axis. For α_i, we can calculate the difference in x and y coordinates of P_{b_i} and P_{a_1} based on their respective L values, then taking the inverse tangent gives us

$$\alpha_i = \arctan\left(\frac{(L_{a_1} + L_{b_i})\sin\theta}{a_1 - b_i - (L_{a_1} - L_{b_i})\cos\theta}\right). \tag{6}$$

Finally, L_{a_1} can be determined by applying Kawasaki's Theorem to P_{b_1} to compute the angle of the crease from P_{b_1} to P_{a_1} as $180^\circ - \theta - |\beta_1|$ (with absolute values because β_1 is in fact negative), then applying Law of Sines to $\triangle P_{b_1}P_{a_1}(a_1,0)$ to obtain

$$L_{a_1} = \frac{\sin(\theta + |\beta_1|)}{\sin|\beta_1|}(a_1 - b_1). \tag{7}$$

3.2 1-to-n Flat Foldability

Next we prove that our 1-to-n transition construction is flat foldable if the alternating-sum condition holds. We start with a base case:

Lemma 2. *Given a* 1*-to-*3 *input satisfying* $a_1 = b_1 - b_2 + b_3$*, there is a locally flat-foldable transition.*

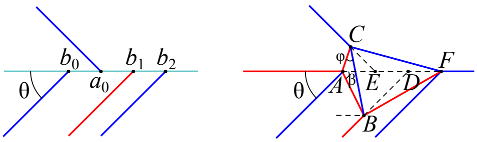

Figure 6: *The* 1*-to-*3 *base case. Vertices* A,B,C,D,E,F *are labeled for convenience.*

See Figure 6. We omit the proof, which consists of several applications of Kawasaki's Theorem, Interior Angle Theorem, and Law of Sines.

Theorem 3. *Given a* 1*-to-*n *input satisfying* $a_1 = b_1 - b_2 + b_3 - \cdots + b_n$*, there is a locally flat-foldable transition.*

Proof. The proof is by induction on n, which must be odd by the parity equation (2) of the alternating-sum condition. The base case $n = 1$ is trivial: by equation (1) of the alternating-sum condition, $a_1 = b_1$, so the transition is a single vertex with reflectional symmetry through the x axis. Lemma 2 proves the base case $n = 3$. For the induction step, assume $n > 3$ and assume the theorem holds for $n - 2$.

We modify the instance by a sequence of two moves:

1. Shift b_{n-2} and b_{n-1} to the right by $b_n - b_{n-1}$, resulting in b'_{n-2} and b'_{n-1}.
2. Remove b'_{n-1} and b_n, which now overlap.

Here we generalize the 1-to-n transition problem to allow overlapping creases (temporarily). By construction, the new input (A, B') satisfies the alternating-sum condition. By induction, it has a locally flat-foldable transition. To extend this transition to the original input, we need to undo the two operations, and prove they preserve local flat foldability:

2. Add overlapping creases with x-intercepts $b'_{n-1} = b_n$.
1. Shift b'_{n-2} and b'_{n-1} to the left by $b_n - b_{n-1}$.

We justify each of these two steps in turn.

Lemma 4. *Given a locally flat-foldable* 1*-to-*$(n-2)$ *transition for* (A, B)*, we can add* $b_{n-1} = b_n$ *to* B *and construct a locally flat-foldable* 1*-to-*n *transition.*[1]

Proof. We need to show that (3) holds for $P_{b_{n-1}}$ and P_{b_n}, (4) holds for P_{a_1}, and $\beta_n = 0$ and $L_{b_n} = 0$ in order to merge back into the rightward infinite subray of the x axis.

[1] We thank Aloysius Ng and David Lee for their ideas and suggestions with regards to this part of the proof.

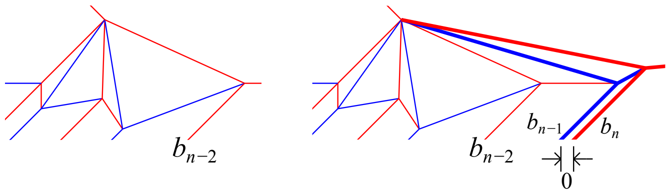

Figure 7: *Left: a* 1*-to-*$(n-2)$ *transition that is given to be flat foldable. Right: adding overlapping creases* $b_{n-1} = b_n$ *preserves flat foldability for any* $b_{n-1} > b_{n-2}$.

Because the given 1-to-$(n-2)$ transition is flat foldable, we know that $L_{b_{n-2}} = 0$ and $\beta_{n-2} = 0$. We can plug these two values into (5) to obtain

$$L_{b_{n-1}} = L_{b_{n-2}} - \frac{\sin\beta_{n-2}}{\sin(\theta - \beta_{n-2})}(b_{n-1} - b_{n-2}) = 0.$$

Then, we plug $L_{b_{n-1}} = 0$ into (6) to obtain

$$\alpha_{n-1} = \arctan\left(\frac{(L_{a_1} + L_{b_{n-1}})\sin\theta}{a_1 - b_{n-1} - (L_{a_1} - L_{b_{n-1}})\cos\theta}\right) = \arctan\left(\frac{L_{a_1}\sin\theta}{a_1 - b_{n-1} - L_{a_1}\cos\theta}\right).$$

Similarly, we can use (5) and (6) to calculate L_{b_n} and α_n:

$$L_{b_n} = \left(L_{b_{n-1}} - \frac{\sin(\beta_{n-1})(b_{n-1} - b_{n-1})}{\sin(\theta - \beta_{n-1})}\right) = 0,$$
$$\alpha_n = \arctan\left(\frac{(L_{a_1} + L_{b_n})\sin\theta}{a_1 - b_{n-1} - (L_{a_1} - L_{b_n})\cos\theta}\right) = \alpha_{n-1}.$$

Next, we use (4) and (3) to prove that all the new vertices in this 1-to-n transition are flat foldable. We know the given 1-to-$(n-2)$ transition was flat foldable, so (4) gives $\alpha_1 - \alpha_2 + \cdots + \alpha_{n-2} = 180^\circ - \theta$. Combining with $\alpha_n = \alpha_{n-1}$ from above, we obtain $\alpha_1 - \alpha_2 + \cdots + \alpha_{n-2} - \alpha_{n-1} + \alpha_n = 180^\circ - \theta$ as desired. Similarly, we know from (3) that $P_{b_{n-1}}$ and P_{b_n} are flat foldable by choosing $\beta_{n-1} = -\beta_{n-1} + \alpha_{n-1} - 180^\circ + \theta$ and $\beta_n = -\beta_{n-1} + \alpha_n - 180^\circ + \theta$, so we obtain

$$\beta_n = -(-\beta_{n-2} + \alpha_{n-1} - 180^\circ + \theta) + \alpha_n - 180^\circ + \theta,$$
$$\text{i.e.,} \quad \beta_n = \beta_{n-2} - \alpha_{n-1} + \alpha_n = \beta_{n-2} = 0.$$

Therefore, the 1-to-n transition is locally flat foldable. □

Lemma 5. *Given a locally flat-foldable* 1*-to-*n *transition, the positions* b_{n-2} *and* b_{n-1} *can be shifted by any amount* Δx *and the transition will still be locally flat foldable, provided that* $b_1 \le b_2 \le \cdots \le b_{n-2} \le b_{n-1} \le b_n$ *remains true.*

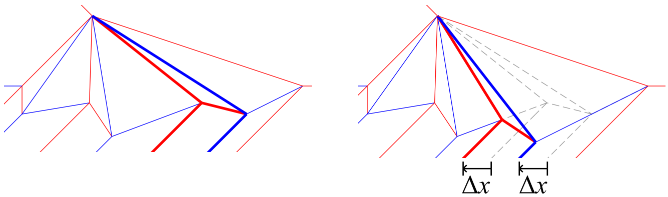

Figure 8: *Left: A flat-foldable* 1*-to-n transition. Right: the same transition but with b_{n-2} and b_{n-1} shifted by some amount Δx. The transition remains flat-foldable because the alternating-sum condition is still true.*

Proof. Because the crease pattern is flat foldable when $\Delta x = 0$, we know that $\beta_n = 0$, which by (3) is satisfied only when

$$\beta_n = -\beta_{n-1} + \alpha_n - 180^\circ + \theta = 0, \qquad \text{i.e.,} \quad \beta_{n-1} = \alpha_n - 180^\circ + \theta.$$

Similarly, using this value of β_{n-1}, we can use (3) on b_{n-1} to find

$$\alpha_n - 180^\circ + \theta = \beta_{n-1} = -\beta_{n-2} + \alpha_{n-1} - 180^\circ + \theta, \qquad \text{i.e.,} \quad \beta_{n-2} = \alpha_{n-1} - \alpha_n.$$

On the other side of β_{n-2}, β_{n-3} does not change with Δx, so we can use (3) on b_{n-2} to find an alternate expression for β_{n-2}:

$$\beta_{n-2} = -\beta_{n-3} + \alpha_n - 180^\circ + \theta.$$

Setting these two expressions for β_{n-2} equal to each other gives

$$\alpha_{n-1} - \alpha_{n-2} = -\beta_{n-3} + \alpha_n - 180^\circ + \theta. \tag{8}$$

This equation indicates that the difference $\alpha_{n-1} - \alpha_{n-2}$ is not dependent on the value of Δx because β_{n-3} and α_n do not change with Δx. Because these are the only two α_i values that are changing, (6) remains true, so a_1 remains flat foldable. We have also shown that β_{n-1} and β_{n-2} remain flat foldable, because we applied (3) to both. We used that the order of creases in B is preserved: otherwise, the validity of (3) breaks down. □

This concludes the proof of Theorem 3. □

3.3 *m*-to-*n* Crease Graph

Now we tackle the general m-to-n transition. As before, let P_{a_i} denote the endpoint of the a_i semi-infinite crease, and let P be the set of all points P_{a_i}, P_{b_i} for all creases in A and B. Then we specify that every crease in the transition will either be one of the parallel input creases in A and B, or a crease that connects two vertices in P. The latter connections form a graph on vertex set P, which we call the ***crease graph***.

To simplify the structure of the graph, we first consider the situation of a ***break*** in A, B, defined as some $i < m$ and $j < n$ where $a_1 - a_2 + a_3 - \cdots a_i = b_1 - b_2 + b_3 - \cdots b_j$. If there is such a break, then A, B up to the break and the A, B after the break can each be

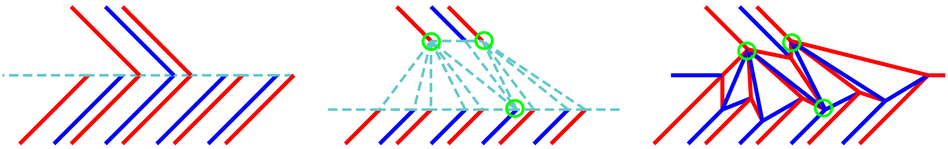

Figure 9: *Left: an example input. Middle: A valid graph based on the input creases. Pivot vertices are highlighted in green. Right: the final crease pattern where each edge of the graph becomes a crease, and vertices have been positioned to become locally flat foldable.*

solved as a separate transition. Similarly, if there is a value x such that adding x to either A or B would make a break at x, then we could also break the input into two transitions by cutting at x. For example, if $A = \{0,1,3,4\}$ and $B = \{1,3\}$, then there exists a value $x = 2$ such that inserting 2 into B would cause a break $0-1+3 = 1-(2)$ and could be solved separately as $\{0,1\},\{1,2\}$ and $\{3,4\},\{2,3\}$. Note that in this case, when the two separate transitions are merged, both sets of B will have a crease at the auxiliary location $x = 2$ and will intentionally cancel out if their mountain valley parity are set to be opposite. We assume in the rest of our construction that there are no such breaks, because if there were, we could solve the two parts separately.

We call a crease graph ***valid*** if there is a locally flat-foldable crease pattern with a crease for each edge in the graph, for some locations for vertices in P. We impose several basic rules for a graph to be valid:

Lemma 6. *A valid crease graph must be outerplanar, with vertices in P appearing on the outside face in a clockwise order matching the clockwise order of rays at infinity.*

Proof. Without outerplanarity, we would have crossings, which would create additional vertices. (Even if we allowed them, the intersection vertices would not be flat foldable.) □

Even stronger, the outerplanar crease graph that we will construct has no cut vertices, so the outer face visits the P_{a_i}s in increasing order followed by the P_{b_j}s in decreasing order. This fact will be a consequence of our construction.

Lemma 7. *Each vertex of the graph must have odd degree, except for the two vertices that connect to the x axis which must have even degree.*

Proof. Each vertex must have an even number of creases on the crease pattern, as an implication of Kawasaki's Theorem [Demaine and O'Rourke 07]. One of these creases will be the vertex's respective semi-infinite input crease, so the number of creases formed by graph connections must be odd. For the two vertices on the x axis, an additional crease will be a semi-infinite x-axis ridge crease, so the number of creases formed by graph connections must be even. □

Lemma 8. *Every vertex must connect to at least one vertex from the opposite set. That is, every P_{a_i} must connect to some P_{b_j}, and every P_{b_j} must connect to some P_{a_i}.*

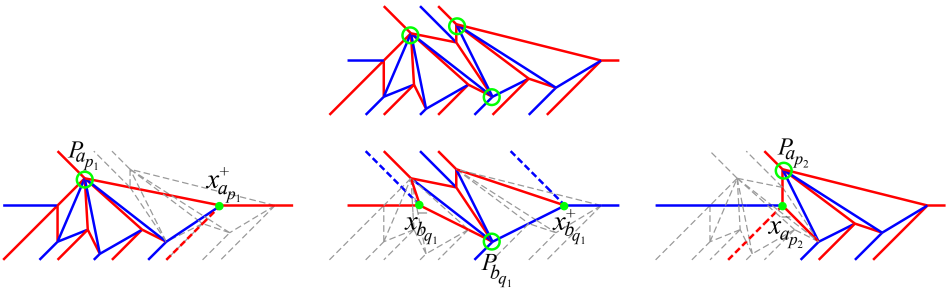

Figure 10: *Top: A complete m-to-n transition. Left, middle, right: The transition can be broken down into a sequence of local* 1*-to-n transitions, where each pivot (circled in green) is the pivot of a local* 1*-to-n transition, each satisfying the alternating-sum condition. Connections to neighboring vertices is replaced with connections to auxiliary* x^- *and* x^+ *points on the x axis.*

Proof. For the sake of contradiction, consider a vertex P_{b_i} that does not connect to any vertices in the opposite set. (The P_{a_i} case is symmetric.) By Lemma 6, this vertex can connect only to its neighbors $P_{b_{i-1}}, P_{b_{i+1}}$ in B. By Lemma 7, it can connect to only one of these neighbors. (If P_{b_i} is at an end, then it has only one neighbor, and we get a contradiction.) Thus P_{b_i} has degree 2. By Kawasaki's Theorem, the two incident creases must be collinear. This cannot happen, assuming the input b_i's are distinct. □

Define a ***pivot*** to be a vertex that is connected to more than one vertex in the opposite set; refer to Figure 9. If a pivot P_{a_i} is connected to vertices $P_{b_j}, \ldots, P_{b_k}$, then by Lemma 6, only P_{b_j} and P_{b_k} can be pivots; the intermediate vertices must be non-pivots connected only to P_{a_i}. We assume that the pivots are connected in a chain, with each pivot connecting to the previous pivot, the next pivot, and all opposite vertices in between. In particular, the pivots alternate between being from A and being from B. The first neighbor of the first pivot and the last neighbor of the last pivot will be the vertices connected to semi-infinite x-axis ridge creases.

For any vertex P_{b_i} connected to its neighbor $P_{b_{i-1}}$, define $x^-_{b_i}$ to be the x-intercept $< b_i$ that P_{b_i} could connect to instead of $P_{b_{i-1}}$ and remain flat foldable; refer to Figure 10. Similarly, if P_{b_i} is connected to $P_{b_{i+1}}$, then $x^+_{b_i}$ is the x-intercept $> b_i$ that P_{b_i} could connect to instead of $P_{b_{i+1}}$ and remain flat foldable. We require in our construction that these x-intercepts exist, enabling us to reduce to 1-in-n subproblems.

Lemma 9. *The first pivot must be the greater of* a_1 *and* b_1.

Proof. First, P_{a_1} and P_{b_1} cannot both be pivots because by definition a pivot connects to more than one vertex from the opposite set, and connecting P_{a_1} to multiple vertices from B and P_{b_1} to multiple vertices from A would violate Lemma 6. It also cannot be the case that neither P_{a_1} nor P_{b_1} is a pivot, because by our connectivity assumption every non-pivot connects to a pivot, which would again cause crossing graph connections.

For the sake of contradiction, consider by symmetry the case where P_{a_1} is the pivot and $a_1 < b_1$. Then P_{a_1} connects to $P_{b_1}, P_{b_2}, \ldots, P_{b_i}, P_{a_2}$ for some $i \geq 2$. Replacing the connection to P_{a_2} with a connection to $(x^+_{a_1}, 0)$, P_{a_1} would then form a local 1-to-n transition with $a_1, b_1, b_2, \ldots, b_i, x^+_{a_1}$, which is flat foldable by Theorem 3 if $a_1 = b_1 - b_2 + b_3 - \cdots - b_i + x^+_{a_1}$. Because $a_1 < b_1$ and $b_1 < b_2 < \cdots < b_i$ is monotonically increasing, the only way to satisfy the alternating-sum condition is if $x^+_{a_1} < b_i$, contradicting our requirement that $x^+_{a_1} > b_1$. Therefore, the first pivot must be the greater of a_1 and b_1. $\square$

Lemma 10. *The indices for pivots from A are either all even or all odd. The indices for pivots from B are the opposite even-odd parity of the indices from A.*

Proof. First we prove that all pivots within A (and symmetrically B) have the same parity. For the sake of contradiction, consider the case where P_{a_i} is a pivot, and the next pivot from A is $P_{a_{i+k}}$ where k is odd. In the chain of pivots, there must be some pivot P_{b_j} in between P_{a_i} and $P_{a_{i+k}}$. Because P_{b_j} is not the first or last pivot, it must be connected to $P_{b_{j-1}}, P_{a_i}, P_{a_{i+1}}, \ldots, P_{a_{i+k}}, P_{b_{j+1}}$, which is an even number of connections because k is odd, contradicting Lemma 7.

Next we prove that the pivots from A have the opposite parity from B. By symmetry, consider the case where P_{a_1} is the first pivot, and the next pivot on the chain is P_{b_j}. Then P_{a_1} connects to $P_{b_1}, P_{b_2}, \ldots, P_{b_j}, P_{a_2}$. By Lemma 7, this must be an odd number of connections, which implies that j is even. Thus some pivot from A has opposite parity (odd) from some pivot from B (even), which by the equality argued above implies that all pivots from A have opposite parity from all pivots from B. $\square$

3.4 *m*-to-*n* Flat Foldability

Theorem 11. *Given an m-to-n input satisfying the alternating-sum condition, there is a locally flat-foldable transition. Furthermore, there is such a transition that satisfies a given β_1 (the angle of the crease from P_{b_1} to P_{b_2}) provided $|\beta_1| \le 180° - \theta$ and the sign of β matches whether B is below or above the ridge line.*

Proof. We follow the graph conditions and pivot structure described in Section 3.3. Let $\{a_{p_1}, \ldots, a_{p_u}\}$ be the set of pivots from A, and $\{b_{q_1}, \ldots, b_{q_v}\}$ be the set of pivot from B, i.e., $\{p_1, \ldots, p_u\}$ and $\{q_1, \ldots, q_v\}$ are the pivots' indices in their respective sets.

The proof is by induction on the number of input crease positions, $m+n$. The base cases are when $m = 1$ or $n = 1$, corresponding to a 1-to-n transition, which is covered by Theorem 3.

For the inductive step, assume $m, n > 1$ and that the theorem holds for all m-to-n transitions with smaller $m+n$. By symmetry, we assume that $a_1 > b_1$, which by Lemma 9 implies that the first pivot is a_1. Refer to Figure 11. At a high level, we split the given m-to-n input into two parts: a 1-to-$(\le n+1)$ input around the first pivot P_{a_1}, and an $(m+1)$-to-$(< n)$ input where we replace the neighbors of P_{a_1} to make it no longer a pivot. We apply Theorem 3 to the 1-to-$(\le n+1)$ part, and apply the induction hypothesis to the $(m+1)$-to-$(< n)$ part. Then we combine the two transitions to argue that the original m-to-n input has a locally flat-foldable transition.

More precisely, we split the input into two parts as follows. We set q_1 to be the largest even index such that

$$a_1 > b_1 - b_2 + b_3 - \cdots + b_{q_1 - 1}. \tag{9}$$

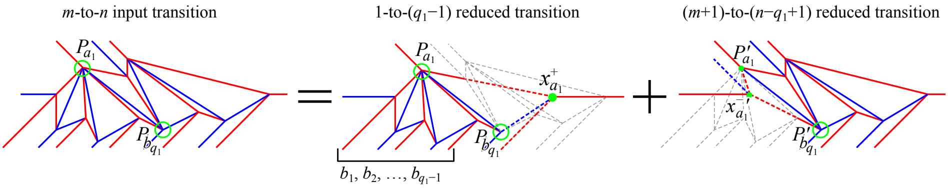

Figure 11: *The reduction operation of our inductive proof. Left: an m-to-n transition. Middle: a local* 1*-to-*(q_1+1) *transition around the first pivot,* P_{a_1}*. Right: the reduced* $(m+1)$*-to-*$(n-q_1+1)$ *transition with one fewer pivot, after replacing* $b_1, b_2, \ldots b_{q_1-1}$ *with their alternating sum* $x^{-\prime}_{a_1}$*. After solving the inputs, we combine the two transitions on the right to form the transition on the left.*

(Assuming every $b_i > 0$, the "even" requirement is in fact implied, because adding a negative term will only reduce the alternating sum.) Note that $q_1 \geq 2$ because $a_1 > b_1$. Then we define two reduced inputs to the transition problem: one is 1-to-(q_1+1) and the other is $(m+1)$-to-$(n-q_1+1)$.

The 1-to-(q_1+1) reduced input consists of a_1 and $b_1, b_2, \ldots, b_{q_1}, x^+_{a_1}$ and the same β_1, where $x^+_{a_1} = a_1 - (b_1 - b_2 + b_3 - \cdots - b_{q_1})$. Thus the alternating-sum condition is satisfied:

$$a_1 = b_1 - b_2 + b_3 - \cdots - b_{q_1} + x^+_{a_1}. \tag{10}$$

Also, this input is correctly in sorted order because $b_{q_1} < x^+_{a_1}$ by equation (9). By Theorem 3, this 1-to-(q_1+1) input has a locally flat-foldable transition. Let $\alpha_1, \alpha_2, \ldots, \alpha_{q_1+1}$ be the angles of the creases incident to P_{a_1} in this transition (in the same order as B).

The $(m+1)$-to-$(n-q_1+1)$ reduced input consists of $B' = \{x^{-\prime}_{a_1}, a_1, a_2, \ldots, a_m\}$, where $x^{-\prime}_{a_1} = b_1 - b_2 + b_3 - \cdots + b_{q_1-1}$, $A' = \{b_{q_1}, b_{q_1+1}, \ldots, b_n\}$, and $\beta'_1 = \alpha_1 - \alpha_2 + \alpha_3 - \cdots + \alpha_{q_1-1}$. (The notation for this input uniformly adds primes.) This input satisfies the alternating-sum condition:

$$\begin{aligned} x^{-\prime}_{a_1} - a_1 + a_2 - a_3 + \cdots \pm a_m &= (b_1 - b_2 + b_3 - \cdots + b_{q_1-1}) - (a_1 - a_2 + a_3 - \cdots \mp a_m) \\ &= (b_1 - b_2 + b_3 - \cdots + b_{q_1-1}) - (b_1 - b_2 + b_3 - \cdots \mp b_n) \\ &= b_{q_1} - b_{q_1+1} + \cdots \pm b_n. \end{aligned}$$

Also, this input is correctly in sorted order because $x^{-\prime}_{a_1} < a_1$ by equation (9). Furthermore, $|\beta'_1| \leq 180° - \theta$ by equation (4).

This "reduced" input has $m+n-q_1+2$ creases, which is $\leq m+n$ because $q_1 \geq 2$. But when $q_1 = 2$, it has the same number $m+n$ of creases as the original input. Fortunately, in this case, we can argue that the *next* such reduction will have strictly fewer creases. First, $x^+_{a_1} > a_2$ because otherwise $x^+_{a_1}$ would be a break: inserting $x^+_{a_1}$ into A would fit between a_1 and a_2, with matching alternating sums by equation (10). The next iteration has $a'_1 = b_{q_1}$, $b'_1 = x^{-\prime}_{a_1}$, $b'_2 = a_1$, and $b'_3 = a_2$, where we have swapped A/B labels so that $a'_1 = b_{q_1} > b_{q_1-1} > b_1 - b_2 + b_3 - \cdots + b_{q_1-1} = x^{-\prime}_{a_1} = b'_1$. Now we claim that the next value q'_1 according to equation (9) will be at least 4 (> 2), i.e., $a'_1 > b'_1 - b'_2 + b'_3$, i.e., $b_{q_1} > x^{-\prime}_{a_1} - a_1 + a_2$: by equation (10), $a_1 = x^{-\prime}_{a_1} - b_{q_1} + x^+_{a_1}$, so by our earlier argument that $x^+_{a_1} > a_2$, $a_1 > x^{-\prime}_{a_1} - b_{q_1} + a_2$.

Thus, we can still effectively apply induction to the $(m+1)$-to-$(n-q_1+1)$ reduced input, and obtain a locally flat-foldable transition. (Technically, we might need to do two reduction steps in a row before $m+n$ decreases and we can apply induction, but the result is the same.)

It remains to show how to combine the two locally flat-foldable transitions of the two reduced inputs to form a locally flat-foldable transition to the original input. Refer again to Figure 11. Our goal is to merge the crease patterns along the segment from P_{a_1} to $P_{b_{q_1}}$, while deleting the dashed segments and the then-isolated horizontal rays (the rightward one in the 1-to-(q_1+1) transition and the leftward one in the $(m+1)$-to-$(n-q_1+1)$ transition). For this to work, we need to check that the geometry of that segment matches in the two diagrams (in particular, that the two versions of P_{a_1} coincide, as do the two versions of $P_{b_{q_1}}$), and that the two merged vertices $P_{a_1}, P_{b_{q_1}}$ remain flat foldable after the merging. (All other vertices come entirely from one of the two reduced transition crease patterns which we know to be locally flat foldable, so they are still locally flat foldable.) In particular, the pivot chain proceeds from P_{a_1} to $P_{b_{q_1}}$ and then as in the $(m+1)$-to-$(n-q_1+1)$ transition.

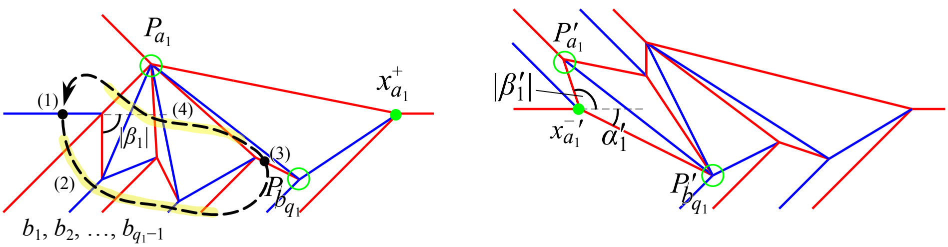

Figure 12: *Left: The* 1*-to-*(q_1+1) *transition, indlucing the closed curve applied with the generalized Kawasaki's Theorem with crossing regions (1) – (4) labelled. Right: The* $(m+1)$*-to-*$(n-q_1+1)$ *transition. We merge the two transitions along the segment from* P_{a_1} *to* $P_{b_{q_1}}$ *such that everything to the left of this segment is constructed according to the* 1*-to-*(q_1+1) *transition, and everything to the right is constructed according to the* $(m+1)$*-to-*$(n-q_1+1)$ *transition. For this operation to be valid, we require that the two vertices* P_{a_1} *and* $P_{b_{q_1}}$ *are in the same position in both configurations, and that the operation preserves their flat foldability.*

We apply the generalized Kawasaki's Theorem [Hull 20, Theorem 6.4] to the 1-to-(q_1+1) transition crease pattern; refer to Figure 12. Specifically, there is a closed curve that crosses (1) the leftward x-axis crease, (2) the $b_1, b_2, \ldots, b_{q_1-1}$ creases in order, (3) the crease connecting $P_{b_{q_1-1}}$ and $P_{b_{q_1}}$, and then (4) the creases connecting P_{a_1} and $P_{b_{q_1-1}}, P_{b_{q_1}-2}, \ldots, P_{b_2}, P_{b_1}$ in order. Thus the correspondence sequence of reflections compose to the identity transformation. The creases in (2) are all parallel, so the composition of their reflections is equivalent to a single reflection, whose line has angle θ and x-intercept $b_1 - b_2 + b_3 - \cdots + b_{q_1-1} = x^{-\prime}_{a_1}$. The creases in (4) are all incident to P_{a_1}, so the composition of their reflections is equivalent to a single reflection, whose line is incident to P_{a_1} and whose angle is $\alpha_1 - \alpha_2 + \alpha_3 - \cdots + \alpha_{q_1-1}$. Thus we obtain four lines whose composed reflection is the identity. It follows that the four lines meet at a point: the composition of the first two or last two reflections is a rotation, so to be inverses of each other, they must share a rotation center.[2] In fact, this intersection point must be $(x_{a_1}^{-\prime}, 0)$, because this is the intersection of lines (1) and (2). Lines (3) and (4) pass through $P_{b_{q_1}}$ and P_{a_1} respectively.

Now look at the $(m+1)$-to-$(n-q_1+1)$ reduced transition crease pattern; refer back to Figure 11. It has a vertex at $(x_{a_1}^{-\prime}, 0)$ (where the four reflection lines meet), with creases at angles $180°$ (leftward), $180° - \theta$, and $\beta_1' = \alpha_1 - \alpha_2 + \alpha_3 - \cdots + \alpha_{q_1-1}$. These creases match the angles of lines (1), (2), and (4), respectively, except that line (2) is reflected through the horizontal ridge line, but such a reflection does not affect flat foldability. Because this vertex is flat foldable in the reduced transition, its fourth crease must match the angle of line (3), i.e., it must pass through $P_{b_{q_1}}$. Because this crease passes through $P'_{b_{q_1}}$, as does the θ input crease with x-intercept b_{q_1}, we conclude that $P_{b_{q_1}} = P'_{b_{q_1}}$, i.e., this vertex is at the same location in both reduced transitions. Similarly, P'_{a_1} is at the intersection of line (4) and the θ input crease with x-intercept a_1, so $P_{a_1} = P'_{a_1}$. Thus the two reduced transition crease patterns agree on the crease connecting P_{a_1} and $P_{b_{q_1}}$, so we can merge the two patterns together along that segment.

Finally, we argue flat foldability of the two merged vertices P_{a_1} and $P_{b_{q_1}}$. In fact, $P_{b_{q_1}}$ has exactly the same incident crease directions in the merged m-to-n transition as in the $(m+1)$-to-$(n-q_1+1)$ reduced transition. Because that vertex is flat foldable in the $(m+1)$-to-$(n-q_1+1)$ reduced transition, so it is in the merged m-to-n transition. To show P_{a_1} is flat foldable in the merged m-to-n transition, it suffices to show that the incident crease to $x_{a_1}^{+}$ in the 1-to-(q_1+1) reduced transition is in the same direction as the incident crease to P_{a_2} in the $(m+1)$-to-$(n-q_1+1)$ reduced transition. This follows because we know P_{a_1} is flat foldable in each reduced transition, so by Kawasaki's Theorem, we know the alternating sum of incident crease angles is zero in both cases. We already know that the θ crease and the crease to $P_{b_{q_1}}$ match in the two transitions, and the creases to $P_{b_1}, P_{b_2}, \ldots, P_{b_{q_1-1}}$ in the 1-to-(q_1+1) reduced transition have the same alternating sum as the crease to $x_{a_1}^{-\prime}$ in the $(m+1)$-to-$(n-q_1+1)$ reduced transition. Thus the one remaining crease directions must match. □

4 Implementation

In this section, we describe an algorithm for finding a desired m-to-n transition, expanding our inductive proof into a linear-time iterative algorithm. An implementation of this algorithm is available online.[3] The inputs are the crease locations in A and B, crease angle θ, and an initial angle β_1. Similar to Section 3.3, we assume that the input has no breaks; if the algorithm finds a break, it re-initializes and solves the input after the break.

4.1 Crease Graph

First, we generate the crease graph by choosing the pivot vertices. In addition to input crease positions a_i, b_j, let $S(a_i), S(b_j)$ denote the alternating sums of their respective sets

[2] We thank Lily Chung for suggesting this proof.

[3] https://github.com/theplantpsychologist/transitions-implementation

up until a_i, b_j; for example, $S(a_i) = a_1 - a_2 \cdots \pm a_i$. We will use rolling indices i_A, i_B to denote the current vertices that are allowed to make connections. Once we connect $P_{a_{i_A}}$ to $P_{b_{i_B}}$, no connections may form with any $P_{a_{i<i_A}}$ or $P_{b_{i<i_B}}$ to avoid crossing connections. Thus, each iteration of the algorithm consists of choosing between two possible actions: connecting $P_{a_{i_A}}$ to $P_{b_{i_B+1}}$, $P_{b_{i_B}}$ to $P_{b_{i_B+1}}$, and increasing i_B by 1 (which we call ***stepping B***), or connecting $P_{b_{i_B}}$ to $P_{a_{i_A+1}}$, $P_{a_{i_A}}$ to $P_{a_{i_A+1}}$, and increasing i_A by 1 (which we call ***stepping A***). The algorithm terminates when $i_A = m$ and $i_B = n$.

The algorithm initializes with $i_A = i_B = 0$. By Lemma 9, the first step (essentially the first pivot) is chosen to be the greater of a_1 and b_1: if $a_1 > b_1$, then we step B, otherwise, we step A. By Lemma 10, all subsequent steps until the last step must step the same side twice in a row to preserve parity. If $|S(a_{i_A+1})| < |S(b_{i_B})|$, then we step A twice. If $|S(a_{i_A})| > |S(b_{i_B+1})|$, then we step B twice. This is equivalent to how we defined q_1 in the proof of Theorem 11. Otherwise, the transition is finished or we hit a break, in which case we decompose the problem.

4.2 Vertex Locations

Once we have the crease graph, we propagate the Kawasaki flat-foldability condition until all vertices have been placed. The procedure initializes by placing the smaller of a_1 and b_1 with $L = 0$. Suppose that $a_1 > b_1$ (if $b_1 > a_1$, the result would be symmetric). We then use the input parameter β_1 to place P_{a_1} and P_{b_i}, by calculating L_{a_1} and L_{b_2} using equations (7) and (5) from Section 3.1. Similar to the crease-graph algorithm, we use two rolling indices i_A and i_B. After initialization, $i_A = 1$ and $i_B = 2$.

Until termination, one of a_{i_A} or b_{i_B} will have k graph connections where $k-1$ of the connected creases already have an assigned position. This is a result of our assumed pivot structure. We then use Kawasaki's Theorem to compute L for the remaining crease such that the current vertex (a_{i_A} or b_{i_B}) becomes flat foldable, then increment the respective rolling index by 1. The algorithm terminates when $i_A = m$ and $i_B = n$.

4.3 Solution Variability

In general, there are many possible flat-foldable transitions between two given sets of input pleats. One obvious source of solution variability is in the input parameter β_1. Some inputs also have multiple valid crease graphs, including beyond the structure we assumed in Section 3.3. Even within our assumed structure, there can be solutions with multiple valid pivot choices.

5 Applications

While our initial motivator for generalized ridge transition units is for use as level shifters in box-pleated or hex-pleated designs, representational design in practice rarely requires such arbitrarily complex level shifters. Designers will rarely use ridge transitions that shift more than 2 or 3 units due to practicality issues such as the space taken up by the transition itself, or the difficulty of folding excessively complicated transitions. We now present an additional application of generalized ridge transitions that could potentially be useful for representational origami design.

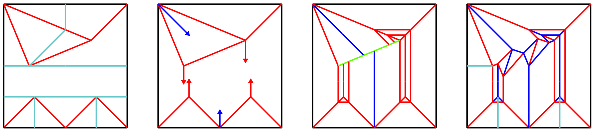

Figure 13: *This algorithm can be used to terminate infinite dense bouncing. From left to right: (1) The initial packing, with ridges shown in red and hinges shown in teal. (2) The bouncing creases begin. (3) Keep bouncing until each crease has hit and stopped at the transition ridge shown in green. (4) Construct a transition around the transition ridge, and adjust the mountain-valley directions. The crease pattern is complete and flat foldable.*

5.1 Terminating Dense Bouncing

Dense bouncing is a problem that arises in uniaxial bases where axial creases will be forced to bounce densely across ridges without termination [Demaine and O'Rourke 07]. Dense bouncing is usually solved by discretizing ridge lengths to a grid system such as box pleating or hex pleating [Lang 11], or in the case of 22.5° designs, by constraining ridge creases into tiles of known molecules.

As shown in Figure 13, it is possible to to terminate dense bouncing in some crease patterns by constructing a generalized ridge transition across a ridge. First, pick one ridge crease to be the "transition ridge", and bounce each axial crease until it either terminates naturally or hits the chosen transition ridge. Then, if the axial creases that touch the transition ridge satisfy the alternating-sum condition, construct a generalized ridge transition around the transition ridge, and the crease pattern will no longer contain dense bouncing.

We conjecture that this procedure always works if there is a closed axial polygon bounding the ridges in question (for Figure 13, the square is the axial polygon) and leave it as an open problem to prove that this is true. The possibility of terminating crease patterns that would otherwise bounce densely opens up new possibilities of fully generalizing uniaxial 22.5° or other non-grid based uniaxial methods.

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