

# Algorithms for Designing Pop-Up Cards

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## Abstract

We prove that every simple polygon can be made as a (2D) pop-up card/book that opens to any desired angle between 0 and 360°. More precisely, given a simple polygon attached to the two walls of the open pop-up, our polynomial-time algorithm subdivides the polygon into a single-degree-of-freedom linkage structure, such that closing the pop-up flattens the linkage without collision. This result solves an open problem of Hara and Sugihara from 2009. We also show how to obtain a more efficient construction for the special case of orthogonal polygons, and how to make 3D orthogonal polyhedra, from pop-ups that open to 90°, 180°, 270°, or 360°.

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## 1 Introduction

Pop-up books have been entertaining children with their playful mechanics since their mass production in the 1970s. But the history of pop-ups is much older [27], and they were originally used for scientific and historical illustrations. The earliest known example of a “movable book” is Matthew Paris’s *Chronica Majora* (c. 1250), which uses turnable disks (volvelle) to represent a calendar and uses flaps to illustrate maps. A more recent scientific example is George Spratt’s *Obstetric Tables* (1850), which uses flaps to illustrate procedures for delivering babies. Dean & Sons’ *Little Red Riding Hood* (1850) is the first known movable book where a flat page rises into a 3D scene, though here it was actuated by pulling a string.

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The first known examples of *self-erecting* pop-ups, where the rise into 3D is actuated by opening the page, are a card promoting the Trinity Buildings in New York City (c. 1908), and S. Louis Girard's *Bookano Book* (c. 1930s). Modern pop-ups have taken these principles to new heights, often employing linkage-like mechanisms to form elaborate 3D shapes and motions; some good guides for designing pop-ups are [1, 3, 5, 20]. In recent years, pop-up books have risen to an art form with such art books as Bataille's *ABC3D* [2], Carter's series of dot/spot books [4], and Pelhem's poetic pop-up book [26]. One striking form of pop-ups is *origamic architecture*, which form buildings and other geometric structures, and are usually made by cutting a single sheet of card stock. A few examples of origamic architecture books are [7, 8, 32]; see [11] for a thorough bibliography.

**Our results.** This paper investigates the computational geometry of pop-ups, in particular, algorithmic design of pop-ups. We achieve three main results:

1. Any 2D  $n$ -gon (extruded orthogonally into 3D) can be popped up by opening a book to a specified angle  $\theta$  with  $0 < \theta \leq 360^\circ$ , using a construction of complexity  $O(n^2)$ .
2. Any orthogonal  $n$ -gon (extruded orthogonally into 3D) can be popped up by opening a book to a specified angle  $\theta \in \{90^\circ, 180^\circ, 270^\circ, 360^\circ\}$ , using a construction of complexity  $\Theta(n)$ .
3. Any orthogonal polyhedron can be popped up by opening a book to a specified orthogonal angle  $\theta \in \{90^\circ, 180^\circ, 270^\circ, 360^\circ\}$ , using a construction of complexity  $O(n^3)$ .

All of our constructions use rigid flat polygonal pieces to form single-degree-of-freedom linkage structures, which uniquely and deterministically unfold from the flat state to the open state, while avoiding collision.

**Related work.** Our results solve an open problem of Hara and Sugihara [14], who gave an algorithmic construction for arbitrary polygons, but with no guarantees of collision avoidance (and indeed the construction sometimes requires collisions). In another result in computational geometry, Uehara and Teramoto [31] proved that pop-ups with creases that can fold both mountain and valley are NP-hard to open or close.

In computer graphics, Mitani et al. [23, 24] showed how to automatically design pop-ups within a common class of  $90^\circ$  origamic architecture, in which the surface is monotone (hit only once) in the view direction. This work led to Tama Software's Pop-Up Card Designer [30]. Li et al. [22] developed a software system for converting a given 3D model into one that fits within this class. Several other systems enable designing and simulating pop-ups by composing standard pop-up gadgets, including Glassner's [12, 13], Popup Workshop [16, 15], Okamura and Igarashi's [25], and Iizuka et al.'s [19].

Geometric pop-ups have also been studied for specific examples of polyhedra. The first such example is a rhombic dodecahedron of the second type [10]. Other examples include the dodecahedron [29] and other Platonic solids [17, 6, 21]. These types of pop-ups are typically not attached to pages of a book, however.

**Applications.** Pop-ups have potential practical applications as well. Nano and micro fabrication technology are well-established for patterning 2D sheets, but remain in their infancy for 3D surfaces. Pop-ups offer a way to transform patterned 2D sheets into 3D surfaces. This idea was recently explored in the context of MEMS [18], where Hui et al. manufactured a 1.8mm-tall 3D model of the UC Berkeley Campanile clock tower using pop-ups.

## 2 Models of Pop-Ups

Our basic model is of a book with planar front and back covers which, when opened to a desired angle  $\theta$ , pops up a 3D paper construction made from pieces of stiff paper that are folded and glued to each other and to the covers. (We will not deal with the more restrictive model of origamic architecture where one piece of paper is cut and folded but not glued.)

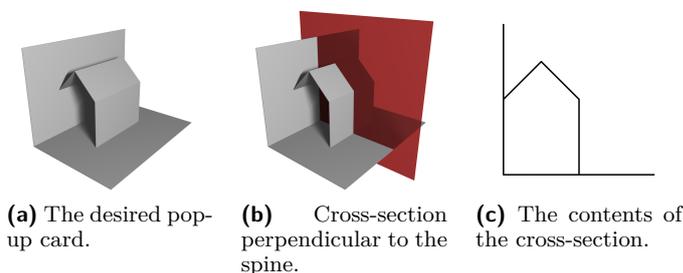


Figure 1 Three views of a desired 3D structure, before the creases and extra paper have been added to make it pop up.

Given a desired 3D structure, we aim to design a book that pops up the structure by adding creases and extra pieces of paper. Adding creases may be necessary to let the structure fold up when the book is closed. Adding extra paper may be necessary to make the structure pop up into the correct shape when the book is opened.

Until Section 5, we consider a restricted version of the problem that arises when all fold lines and all gluing lines are parallel to the spine, as in Figure 1. In this case, a cross-section of the 3D structure in a plane perpendicular to the spine yields a 2D pop-up: the pop-up structure forms a planar linkage composed of rigid bars (line segments) connected at joints. A joint is a point where bars intersect, usually at an endpoint of at least one of the bars. We distinguish three kinds of joints:

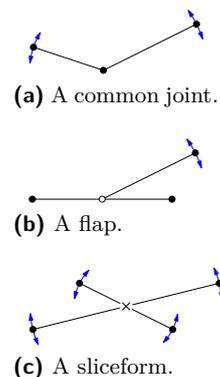


Figure 2 The three types of joints used in this paper.

**Common joints:** Two or more bars are linked at one of their endpoints.

**Flaps:** A bar contains a joint in its interior, where an endpoint of another bar is linked. The location of the joint at the interior of the first bar is fixed.

**Sliceforms:** A joint (called a sliceform) can be formed by the intersection  $X$  of two bars. The intersection point  $X$  cannot shift along the bars, but the two bars can change their angle at  $X$  (scissors-like). Notice that we do not consider the two edges crossing if they are linked by a sliceform.

To distinguish the different joints in figures, we use a dot ( $\bullet$ ) for common joints and endpoints of edges, an empty circle ( $\circ$ ) for flaps, and a cross ( $\times$ ) for sliceforms.

The common joint is sufficient to simulate the other joint types.

A flap can be simulated by forming a zero-area triangle among the three collinear points. A sliceform can be simulated by common joints and flaps as illustrated in Figure 3.

In the 2D case, we want to construct a linkage  $L$  with one degree of freedom that unfolds to the desired polygon  $P$ . During the folding motion we require that no bars cross, and that the order of the bars emanating from a joint is preserved. Let the vertices of  $P$  be  $v_1, v_2, \dots, v_n$  labelled in counter-clockwise order. The edge incident to  $v_i$  and  $v_{i+1}$  is named  $e_i$ , and the edge between  $v_n$  and  $v_1$  is named  $e_n$ . We assume that  $P$  is contained in one of the two

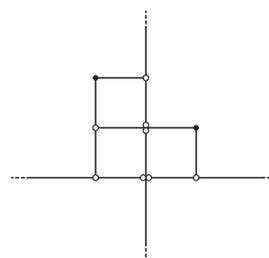


Figure 3 Simulating sliceforms.

wedges bounded by the rays  $\overrightarrow{v_1v_2}$  and  $\overrightarrow{v_1v_n}$ . The angle of the wedge containing  $P$  is called the *opening angle*, and the union of the rays  $\overrightarrow{v_1v_2}$  and  $\overrightarrow{v_1v_n}$  is called the *cover*. We require  $L$  to have the following properties:

1. In one configuration of  $L$ , the boundary of  $L$  coincides with  $P$ . We call this the *open configuration*. The linkage  $L$  contains the edges  $e_1$  and  $e_n$  of  $P$  as bars. If a joint of  $L$  coincides with a vertex  $v_i$  in the open configuration, we name it  $p_i$ .
2. In one configuration of  $L$  that can be reached from the open configuration, all edges are collinear and  $p_1$  is an endpoint of the union of the edges of  $L$ . This configuration is called the *closed configuration*.
3. There is a unique motion that transforms the open configuration into the closed configuration. During this motion,  $L$  is contained inside the wedge defined by the cover and the opening angle decreases continuously. We refer to this motion as the *closing motion*. Every configuration of  $L$  obtained during the closing motion is called an *intermediate configuration*. The open configuration might have several joints that are opened  $180^\circ$ . In order to specify the folding uniquely, we prescribe for every such ambiguity the way the vertex moves during the folding motion. Collinear points in the open configuration appear naturally in pop-up structures. In the real world the folding motion at these points is prescribed by the creasing of the paper.

The *combinatorial complexity* of a 2D pop-up is equal to the number of joints in the pop-up.

### 3 Orthogonal Polygon Pop-Ups

In this section, we assume the polygon  $P$  is orthogonal, i.e., every edge of  $P$  is either horizontal or vertical. Under this assumption, we show how to construct a pop-up linkage  $L$  for the polygon  $P$  with combinatorial complexity linear in  $n$ . The techniques we use in this section are based on a particular type of motion:

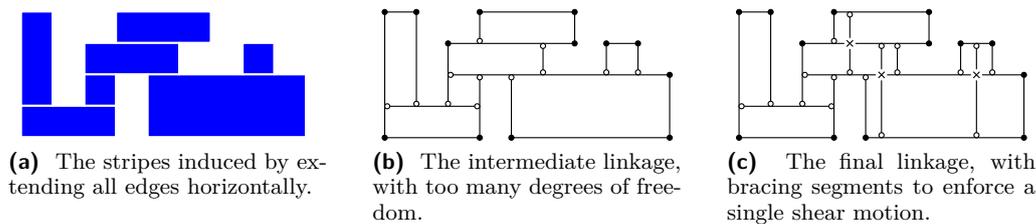
► **Definition 1.** A *shear* is a motion of a linkage that leaves parallel edges parallel.

In Section 3.1, we explain how to construct pop-ups for polygons with opening angle  $90^\circ$ , also known as *90° pop-ups*. In Section 3.2, we extend this result to larger opening angles.

#### 3.1 90° Pop-Ups

To construct  $90^\circ$  pop-ups, we use a process called *h-superimposing*. As a first step we split  $P$  into *stripes* such that (i) each stripe is an axis-aligned rectangle, (ii) the left and right boundary edges of a stripe are a part of the boundary of  $P$ , and (iii) the union of any two stripes is not a rectangle. We obtain such a decomposition by extending all horizontal edges of  $P$  horizontally as long as they lie in  $P$ . See Figure 4a for an illustration. Two stripes are *adjacent* if they (partially) share an edge.

Let  $L_1$  be the linkage obtained by extending all horizontal edges as long as they lie within  $P$ . The newly introduced degree-3 vertices become flaps. An example of this is depicted in Figure 4b. This intermediate linkage may have more than one degree of freedom: any pair of adjacent stripes that do not share a vertical bar can shear independently. To handle this, note that for any pair of adjacent stripes, there must be at least one vertical line passing through the (strict) interior of both stripes. We call this a *bracing line* for the stripe pair. The subset of the line contained in the stripe pair is called a *bracing segment*. For each pair of adjacent stripes that do not share a vertical bar, we add a bracing segment to the linkage, creating a sliceform joint where the segment intersects with the boundary between



■ **Figure 4** The result of h-superimposing an orthogonal polygon.

the stripes. See Figure 4c for an example. Let  $L_2$  be the linkage resulting from the addition of the bracing segments.

► **Theorem 1.** The linkage  $L_2$  obtained by h-superimposing defines a pop-up fold for the orthogonal polygon  $P$  with  $90^\circ$  opening angle. The motion of  $L_2$  is a shear. The combinatorial complexity of  $L_2$  is  $O(n)$ .

All omitted proofs may be found in the full version of this paper.

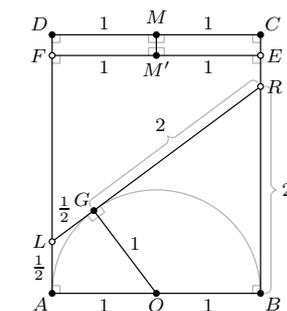
### 3.2 180°, 270°, and 360° Pop-Ups

This section is devoted to constructing pop-up folds with larger opening angles. We reduce this problem to the  $90^\circ$  pop-up scenario by introducing a linkage (called a *reflector gadget*) that allows us to reflect a shear. The open configuration is constructed as shown in Figure 5a. Figure 5b depicts an intermediate configuration.

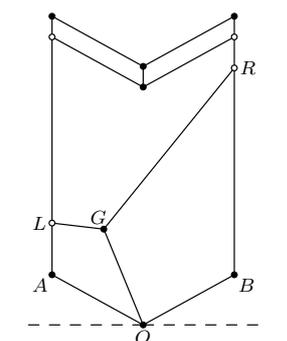
► **Lemma 1.** The reflector gadget has one degree of freedom. Its closing motion has the following properties:

1. the vertical line segments in the open configuration remain vertical during the induced motion,
2. the boundary of the gadget is symmetric with respect to a line of reflection running through  $\overline{OM}$ , and
3. the linkage folds to a line without introducing any crossings in an intermediate configuration.

We use the properties of the reflector to combine two  $90^\circ$  pop-ups to create a pop-up with larger opening angle. We discuss  $180^\circ$  pop-ups first. In this case both cover edges lie on a line through  $p_1$ . To guide our construction we add a bisector  $s$  of the cover edges that runs through  $p_1$ . Furthermore, we add two lines parallel to  $s$  such that the induced stripe contains  $s$  and does not contain any point of  $P$  except those lying on  $s$ . This stripe is called  $\mathcal{S}$ . The edges that “appear” when intersecting  $P$  with the boundary of  $\mathcal{S}$  are added to the linkage  $L$ . We “fill” each rectangle obtained by intersecting  $P$  with  $\mathcal{S}$  with a reflector gadget. The components of  $P \setminus \mathcal{S}$  are turned into a linkage by h-superimposing as discussed in Section 3.1, so that every component of  $P \setminus \mathcal{S}$  supports a shearing motion. The shearing motions are linked by the reflector gadgets, so the combined linkage  $L$  has one degree of freedom. By the properties of the reflector, the left and right side of

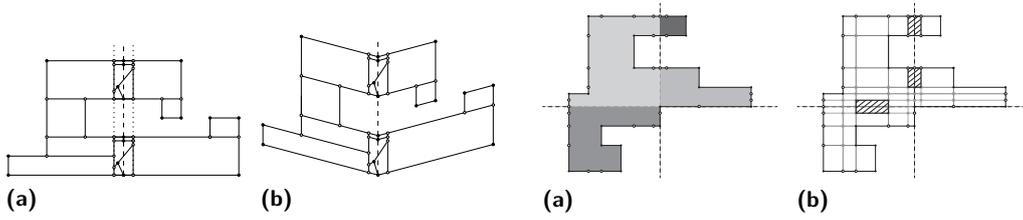


(a) The open configuration.



(b) An intermediate configuration.

■ **Figure 5** The reflector gadget that helps to “reflect” two shearing motions.



■ **Figure 6** A  $180^\circ$  pop-up fold constructed with the help of reflector gadgets. (a) The open configuration. (b) An intermediate configuration.

■ **Figure 7** (a) A polygon with opening angle  $270^\circ$ . The induced connected components are drawn with different shades of grey. (b) The pop-up linkage. The reflector gadgets have to be inserted at the crossed regions.

$s$  perform a shear and both parts of  $P$  stay on their own side, relative to  $s$ . Hence it is impossible for  $L$  to self-intersect. Notice that we can always make the stripe  $\mathcal{S}$  thin enough that the rectangles of  $P \cap \mathcal{S}$  are not “too wide” for the reflector gadgets. See Figure 6 for an example. We conclude with:

► **Theorem 2.** The method described above constructs a pop-up fold for the polygon  $P$  with opening angle  $180^\circ$ . The combinatorial complexity of the linkage is  $O(n)$ .

In order to realize  $270^\circ$  and  $360^\circ$  folds we extend the  $180^\circ$  construction as follows. We split  $P$  into pieces by cutting it along the horizontal and vertical lines through  $p_1$ . We then turn each connected component of the split polygon into a  $90^\circ$  linkage, by adding bars and joints as discussed in Theorem 1. Then each piece of the polygon will be constrained to move in a shear motion, but different pieces will not necessarily move together. To synchronize the pieces, we use reflector gadgets to connect them. To generate the space for the gadgets, we add bars that sandwich the horizontal and vertical lines through  $p_1$ , thereby creating vertical and horizontal strips in which the reflector gadgets can be placed. Because no gadgets lie inside the intersection of the vertical and horizontal strip, no two reflector gadgets interfere. Figure 7 shows an example of an  $270^\circ$  fold. We conclude with the following theorem:

► **Theorem 3.** The method described above constructs a pop-up fold for the polygon  $P$  with opening angle  $270^\circ$  or  $360^\circ$ . The combinatorial complexity of the linkage is  $O(n)$ .

## 4 General Polygon Pop-Ups

In this section we provide a different method for constructing pop-ups of polygons. This method works for all simple  $P$  (not necessarily orthogonal), but has a higher asymptotic complexity of  $O(n^2)$ . Before giving the construction, we provide a key geometric lemma about the non-crossing of nested “V-fold” linkages.

### 4.1 Nested V-folds

We define an *outward V-fold* as the single-degree-of-freedom linkage formed by a (weakly) convex quadrilateral  $ABCD$  with  $AB + BC = AD + DC$ . (This was called a V-fold in [14].) Such a linkage folds flat as the opening angle  $\angle BAD$  decreases to zero. If, in the open configuration, the angle at  $C$  is  $180^\circ$  and the angle at  $A$  is less than  $180^\circ$  (i.e. the quadrilateral is a nontrivial triangle with  $C$  on side  $BD$ ), we call this linkage a *flat outward V-fold*. Similarly, the linkage formed by a (weakly) non-convex quadrilateral  $ABCD$  with

$AB - BC = AD - DC$  has one degree of freedom and folds flat without overlap, and is called an *inward V-fold*. If the angle at  $C$  is  $180^\circ$  and the angle at  $A$  is less than  $180^\circ$  it is a *flat inward V-fold*.

► **Theorem 4.** (a) Let  $ABCD$  and  $AB'C'D'$  be flat outward V-folds on the same rays with  $\triangle BAD \subset \triangle B'AD'$ , where we may have  $B = B'$  or  $D = D'$ . Then these linkages do not cross during the closing motions. In fact, they do not touch at all, except at the closing configuration and possibly at the endpoints  $B = B'$  or  $D = D'$  if either equality holds.

(b) The same statement holds with “outward” replaced by “inward”.

## 4.2 The General Pop-Up Construction: The Method

We may now describe the construction for pop-ups of general polygons. As in Section 2, we wish to construct a one-degree-of-freedom linkage  $L$  contained in simple polygon  $P = v_1v_2 \cdots v_n$ , where  $P$  is contained in the wedge formed by rays  $v_1v_2$  and  $v_1v_n$ . We sometimes refer to the crease point  $v_1$  as  $O$ . The opening angle  $\theta$  of the original configuration, namely the angle of polygon  $P$  at vertex  $O$ , may take any value  $0 < \theta \leq 360^\circ$ .

First we discuss the general strategy and provide a linkage  $L_1$  that has a pop-up motion for polygon  $P$  but has more than one degree of freedom. Later we brace the linkage to remove the excess flexibility.

We first subdivide the wedge around  $O$  containing  $P$  by rays starting at  $O$ , where there is one such ray through each vertex of  $P$  and additional rays are inserted so that consecutive rays form acute angles. Suppose  $r_1, \dots, r_t$  are these rays in order around  $O = v_1$ , starting at  $r_1 = \overrightarrow{Ov_2}$  and ending at  $r_t = \overrightarrow{Ov_n}$ . The region of the plane between rays  $r_i$  and  $r_{i+1}$  is the  *$i$ th wedge*,  $W_i$ . We subdivide polygon  $P$  by these rays: any positive length segment of a ray  $r_i$  contained in  $P$  or its boundary is inserted as a single bar in linkage  $L_1$  and is called a *wall segment*. Notice that edges of  $P$  may be wall segments. Also, by slight abuse of terminology, a positive length subsegment of a wall segment is also called a wall segment. Any isolated points on  $r_i \cap P$  are necessarily vertices of  $P$  and are called *wall points*.

The rays  $r_i$  subdivide  $P$  into a number of triangles and quadrilaterals, called *cells*. Each cell has two wall portions on consecutive rays: at least one of these is a wall segment, and the other may be a wall segment or point. A cell that has two wall segments is called an *internal cell*, and those with a wall point are *ear cells*. Two cells are *adjacent* if they share a wall segment. By adding at most one new ray for each ear cell, (and renumbering the rays as necessary), we may assume that each ear cell is adjacent to a unique interior cell.

The rays  $r_i$  also subdivide the boundary of  $P$  into segments. On each such segment  $AB$  that is not a wall segment (which implies  $A$  and  $B$  are on consecutive rays), insert a joint  $C$  at the point that would make  $OACB$  an outward V-fold at  $O$ , i.e.,  $C$  is the unique point on  $AB$  with  $OA + AC = OB + BC$ . This linkage  $L_1$  serves our first stated purpose:

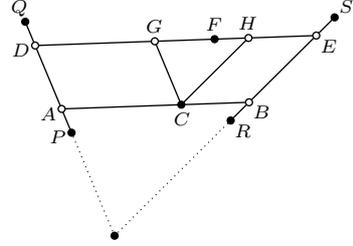
► **Lemma 2.** The linkage  $L_1$ , constructed from  $P$  by adding wall segments and extra boundary vertices as described here, can be continuously folded flat without overlap.

**Proof.** Let  $\phi_i$  be the angle of wedge  $W_i$ , i.e., the angle between rays  $r_i$  and  $r_{i+1}$  at  $O$ . Consider any continuous rotation of rays  $r_1, \dots, r_t$  around  $O$  such that all angles  $\phi_1, \dots, \phi_{t-1}$  decrease monotonically to 0. Let each wall portion on ray  $r_i$  rotate around  $O$  to stay on ray  $r_i$ , and for each boundary portion  $ACB$  of  $P$  within wedge  $W_i$ , let  $ACB$  fold outward as would the outward V-fold  $OACB$ . Then path  $ACB$  remains inside wedge  $W_i$  throughout the motion, and therefore does not interact with portions of  $P$  in different wedges. Furthermore, by Theorem 4, the various boundary portions in wedge  $W_i$  do not touch each other throughout the motion. It follows that this is indeed a continuous planar motion of  $L_1$ . ◀

The rest of the construction shows how to add additional support to  $L_1$  to turn it into a one-degree-of-freedom linkage whose motion has the form described in the proof of Lemma 2. We cut down the freedoms of  $L_1$  in several steps, given in the next three subsections.

### 4.3 Constraining Wall Segments to Rotations

For two segments  $PQ$  and  $RS$  whose lines intersect at a point  $O$ , consider the *rotation gadget* as illustrated in Figure 8. (When we apply this below,  $PQ$  and  $RS$  will be wall segments, and  $O$  will be the crease point.) This linkage is specified as follows:  $AB \parallel DE$  are any two segments not sharing an endpoint with  $PQ$  or  $RS$  with  $AB$  closer to  $O$  than  $DE$ ;  $C$  is chosen on  $AB$  so that  $OA + AC = OB + BC$ , and the  $180^\circ$  angle at  $C$  is declared to fold outward, with  $F$  on  $DE$  chosen similarly;  $G$  and  $H$  are chosen so that both  $DACG$  and  $CBEH$  are parallelograms.



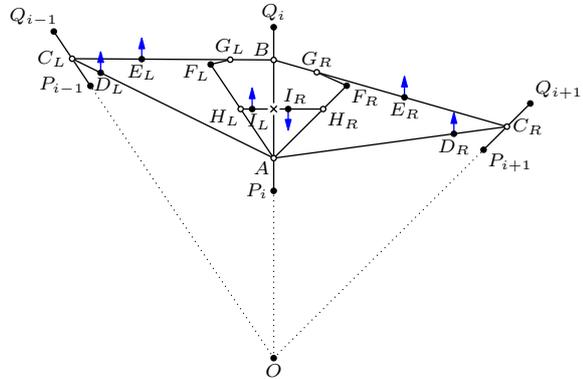
■ **Figure 8** Rotation gadget.

► **Lemma 3.** The linkage illustrated in Figure 8 has one degree of freedom. If  $PQ$  and point  $O$  are held fixed in the plane, then in the unique motion, segment  $RS$  rotates rigidly around point  $O$  from its starting position to a closed configuration where  $PQ$  and  $RS$  are collinear.

► **Lemma 4.** Let  $L_2$  be the linkage derived from  $L_1$  as follows: for every internal cell, attach a rotator gadget inside the cell connecting (internal subintervals of) the wall segments. Then the motions of  $L_2$  correspond exactly to those motions of  $L_1$  where wall segments only rotate around  $O$ , and planar motions of  $L_1$  extend (uniquely) to planar motions of  $L_2$ .

### 4.4 Synchronizing Wall Segments

We next show how to synchronize the wall segments to ensure that all wall segments originally on ray  $r_i$  remain on a single ray through  $O$  throughout any continuous motion. Let  $\phi_1, \dots, \phi_{t-1}$  be the initial angles of the wedges  $W_1, \dots, W_{t-1}$ . For an internal cell  $ABCD$  with  $AB \subset r_i$  and  $CD \subset r_{i+1}$ , we know that any motion of  $L_2$  rotates  $AB$  and  $CD$  around  $O$ , and we define the *angle* of the cell at any time as the angle between rays  $OAB$  and  $OCD$ .



► **Definition 2.** For each  $1 \leq i \leq t - 2$ , ■ **Figure 9** Synchronizing gadget.

construct a linkage  $M_i$  with two adjacent flat V-folds as follows. Points  $A, D, B, E, C$  are collinear, and connected in order (with  $B$  a flap on bar  $DE$ ), and point  $O$  connects to  $A, B$ , and  $C$ . Angle  $OBA$  is  $90^\circ$ ,  $\angle AOB = \phi_i$ , and  $\angle BOC = \phi_{i+1}$ . Finally, if  $i$  is even then  $OADB$  is an *outward* flat V-fold and  $OBEC$  is an *inward* flat V-fold, and if  $i$  is odd then  $OADB$  is chosen outward and  $OBEC$  is inward.

► **Lemma 5.** The linkage  $M_i$  defined as above has a single degree of freedom and folds from the initial configuration to a flat one without overlap. Furthermore, there is a continuous, strictly increasing, and invertible function  $m_i : [0, \phi_i] \rightarrow [0, \phi_{i+1}]$  such that  $m_i(\angle AOB) = \angle BOC$  during this motion.

Inductively define  $\Phi_1(s) = s$  and  $\Phi_i(s) = m_{i-1}(\Phi_{i-1}(s))$ ; these will control the rates at which internal cells' angles change. Specifically, fix an internal cell  $X_1Y_1Y_2X_2$  with two wall segments  $X_1Y_1$  and  $X_2Y_2$  such that  $X_1Y_1 \subset r_1$  and  $X_2Y_2 \subset r_2$  initially. (We may have  $X_1 = X_2 = O$ .) Let  $s$  be a variable representing the angle of cell  $X_1Y_1Y_2X_2$  during any motion. We will brace  $L_2$  to a new linkage so that every internal cell initially in  $W_i$  will have angle  $\Phi_i(s)$  during the motion.

To do this, we make the following additions to  $L_2$  to form a new linkage  $L_3$ : For every pair of adjacent internal cells with wall segments  $P_{i-1}Q_{i-1} \subset r_{i-1}$ ,  $P_iQ_i \subset r_i$ , and  $P_{i+1}Q_{i+1} \subset r_{i+1}$  (note that  $P_iQ_i$  need not be a maximal wall segment for either cell), attach a *synchronizing gadget* as shown in Figure 9. The full version of this paper provides a more detailed description of this process.

► **Lemma 6.** Define  $L_3$  as the linkage constructed from  $L_2$  by inserting a synchronizing gadget between every pair of adjacent internal cells as described above. Then the continuous motions of  $L_3$  correspond to those motions of  $L_2$  such that the angle of any internal cell originally in wedge  $W_i$  is now  $\Phi_i(s)$ , where  $s$  represents the (changing) angle of cell  $X_1Y_1Y_2X_2$ . Furthermore, planar motions of  $L_2$  induce planar motions of  $L_3$ .

## 4.5 Constraining Ear Cells

The configurations of all internal cells in  $L_3$  are determined by  $s = \angle Y_1OY_2$ . The only unwanted degrees of freedom of  $L_3$  must therefore come from the ear cells, which have not yet been modified. In this section we constrain these to produce the final linkage  $L$ .

Consider an ear cell with wall segment  $P_iQ_i \in r_i$  and wall point  $V_{i+1} \in r_{i+1}$ , say. This is adjacent to a unique interior cell, with wall segment  $P_{i-1}Q_{i-1}$  along  $r_{i-1}$ . To constrain ear cell  $P_iQ_iP_{i+1}$ , we simply add *two* synchronization gadgets centered on  $P_iQ_i$  that both connect to  $V_{i+1} \in r_{i+1}$  and some point  $V_{i-1} \in P_{i-1}Q_{i-1}$ . Adding these synchronization gadgets for each ear cell produces the final linkage  $L$ :

► **Theorem 5.** The linkage  $L$  obtained from  $L_3$  by adding two synchronization gadgets to each ear cell is a pop-up for the polygon  $P$ . Its boundary is connected and forms the polygon  $P$  in its opened configuration, and there are  $O(n^2)$  total bars in the linkage.

## 5 Orthogonal Polyhedra Pop-Ups

In this section, we apply some of the techniques of 2D pop-up folds to the design of 3D pop-up structures that take the shape of orthogonal polyhedra. We first show how to construct pop-ups with an opening angle of  $90^\circ$ , then extend the construction to larger opening angles.

### 5.1 3D Pop-Up Model

In the 3D case, we model a pop-up using a model similar to *rigid origami*. A structure in rigid origami is composed of infinitely thin rigid sheets of paper, each in the shape of a simple polygon, connected using hinged joints. If two or more sheets are joined at a hinge and one is held fixed, then the only possible motion for the other sheet(s) is rotation around the hinge. A fold or a crease in a pop-up is equivalent to a hinge connecting two sheets. A flap in a pop-up corresponds to attaching the edge of one sheet to the center of another.

Let  $P$  be a simple polyhedron with  $n$  vertices  $v_1, \dots, v_n$ . We select one edge  $e$  in  $P$  to be the *spine* of the pop-up. Let  $f_1$  and  $f_2$  be the faces adjacent to  $e$ . The *opening angle* of the pop-up is the measure of the dihedral angle between  $f_1$  and  $f_2$ . The *cover* of the pop-up consists of the union of two halfplanes. The first halfplane in the cover is the half of the

supporting plane of  $f_1$  that contains  $f_1$  and has the extension of  $e$  as its boundary. The half of the cover containing  $f_2$  is defined similarly.

A rigid-origami structure  $L$  is a *3D pop-up* for  $P$  if it has an open configuration, a closed configuration, and a unique folding motion from open to closed, all defined analogously to the configurations of a 2D pop-up. The *combinatorial complexity* of the 3D pop-up  $L$  is equal to the number of hinges.

Note that unlike in the 2D case, it is not sufficient to add more paper and more creases. By the Bellows Theorem [28, 9], if we treat a polyhedron as a linkage where each face is rigid and faces must rotate around edges, then all motions preserve the volume of the polyhedron. Hence, we cannot fold the polyhedron flat unless we cut the boundary of the polyhedron.

## 5.2 Scaffold Pop-Ups

Suppose we have a simple orthogonal polyhedron  $P$  with an opening angle of  $90^\circ$ . Without loss of generality, we may assume that  $e$  lies along the  $z$ -axis, and that  $f_1$  lies in the positive  $x$  section of the  $xz$  plane. Suppose further that  $f_2$  lies in the positive  $y$  section of the  $yz$  plane. Let  $x_1, \dots, x_n$  be the sorted  $x$ -coordinates of all vertices in  $P$ . Similarly, let  $y_1, \dots, y_n$  be the sorted  $y$ -coordinates and let  $z_1, \dots, z_n$  be the sorted  $z$ -coordinates. Then *grid cell*  $(i, j, k)$  is the rectangular box  $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$ . By construction, the polyhedron  $P$  is the union of a face-connected subset  $R$  of grid cells. The *scaffold* of  $P$  is the union of all faces  $f$  of cells in  $R$  such that  $f$  is parallel to the spine.

The *grid slice*  $G_k$  consists of the union of all grid cells  $(i, j, k)$ , not necessarily contained in  $P$ . Let the *slice scaffold*  $S_k$  be the intersection of the scaffold with  $G_k$ . The slice scaffold contains no faces perpendicular to the  $z$ -axis, and every cross section perpendicular to the  $z$ -axis is the same. Hence, the problem of constructing a pop-up for  $S_k$  is purely 2D.

To construct a pop-up for  $S_k$  with the correct shear motion, we must somehow combine faces of  $S_k$  into larger rigid sheets. If an edge borders exactly three faces, then the two coplanar faces can be fused into a rigid sheet, with the third face added as a flap. Suppose instead that we have an edge with  $x$  and  $y$  coordinates  $(x_i, y_j)$  bordering exactly four faces. If  $(i + j)$  is even, then we rigidify the pair of faces perpendicular to the  $x$ -axis; otherwise, we rigidify the pair of faces perpendicular to the  $y$ -axis. This construction means that the four sheets adjacent to a given grid cell are arrayed in a pinwheel pattern. This ensures that the shear motion of one cell must be the same as the shear of all adjacent cells.

Suppose that we use this construction to make a pop-up-like structure for each slice, which we will call a *pinwheel slice*. Place all pinwheel slices side-by-side so that the initial position takes the shape of the scaffold. Call the result of this process the *sliced pinwheel scaffold*. Unfortunately, the sliced pinwheel scaffold has too many degrees of freedom: each slice scaffold is disconnected from its neighbors, and even within a single slice the scaffold may be disconnected.

Given any pair  $r_1, r_2 \in R$  of adjacent cells in adjacent slices, we wish to cause any motions of the sheets around  $r_1$  to affect the sheets around  $r_2$ . For each such pair  $r_1, r_2$ , we fuse each of the four sheets that surround  $r_1$  in the initial configuration with the corresponding coplanar sheet around  $r_2$ , to create four larger rigid sheets in the initial opening configuration. Call the result of this fusing the *pinwheel scaffold* of  $P$ .

► **Lemma 7.** The pinwheel scaffold of a polyhedron  $P$  is a pop-up for the scaffold of  $P$ . The pinwheel scaffold has complexity  $O(n^3)$ .

The pinwheel scaffold has a number of faces parallel to the spine. All such faces are contained within  $P$  when the scaffolding is open, and all faces on the boundary of  $P$  that

are parallel to the spine also exist in the scaffolding (although they may be subdivided). The only missing pieces are the faces of  $P$  that are perpendicular to the spine.

### 5.3 Additional Faces

To add those pieces to the pinwheel scaffold, we first subdivide the faces using our rectilinear grid so that the sheets we wish to add to the pinwheel scaffold are faces of the grid cells. We must attach each such sheet to the sheets in the scaffold that form the adjacent grid cell.

There are four potential hinges that we could use to attach the new face to the scaffold. The hinges we choose to use are the hinge parallel to the  $x$ -axis with the smallest  $y$ -coordinate, and the hinge parallel to the  $y$ -axis with the smallest  $x$ -coordinate. By construction, the angle between these two hinges will grow smaller as the pinwheel scaffold folds. Therefore, if we attach the new face using these hinges, it is sufficient to add a crease to the new sheet emanating from the intersection of the two hinges at a  $45^\circ$  angle. For consistency, we make each crease constructed in this fashion fold in the positive  $z$ -direction. We call the resulting rigid origami structure the *draped scaffold*.

► **Theorem 6.** The draped scaffold of  $P$  is a pop-up for  $P$  with complexity  $O(n^3)$ .

The draped scaffold may be used to construct  $90^\circ$  pop-ups in 3D. By combining this structure with a reflector gadget as in Section 3.2, we can extend our construction to handle larger multiples of  $90^\circ$ . See the full version for details.

## 6 Conclusion and Open Problems

In this paper, we demonstrate techniques for designing 2D pop-ups for general polygons, and 3D pop-ups for orthogonal polyhedra. The most obvious open question is whether there is a way to construct 3D pop-ups for general polyhedra. Another question to consider is which 2D or 3D shapes are constructible using a single sheet of material with no gluing, as in most origamic architecture.

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### References

- 1 Carol Barton. *The Pocket Paper Engineer: How to Make Pop-Ups Step-by-Step*. Popular Kinetics Press, Glen Echo, Maryland, 2005–2008. Two volumes.
- 2 Marion Bataille. *ABC3D*. Roaring Brook Press, 2008.
- 3 Duncan Birmingham. *Pop-Up Design and Paper Mechanics: How to Make Folding Paper Sculpture*. Guild of Master Craftsman Publications, 2010.
- 4 David A. Carter. *One Red Dot: A Pop-up Book for Children of All Ages*. Little Simon, 2005. Other books include *Blue 2* (2006) and *600 Black Spots* (2007).
- 5 David A. Carter and James Diaz. *The Elements of Pop-Up*. Little Simon, New York, 1999.
- 6 David Cassell. Pop-up polyhedra. *Mathematics in School*, 17(1):24–27, January 1988.
- 7 Masahiro Chatani. *Key to Origamic Architecture*. Shokokusha, 1985.
- 8 Masahiro Chatani. *Pattern Sheets of Origamic Architecture*. Books Nippan, 1986. Two volumes.
- 9 R. Connelly, I. Sabitov, and A. Walz. The bellows conjecture. *Beiträge Algebra Geom*, 38(1):1–10, 1997.
- 10 John Lodge Cowley. *Solid Geometry*. London, 1752.
- 11 Evermore Origamic Architecture. Pop-up card books. <http://www.evermore.com/oa/books.php3>, 2011.

- 12 Andrew Glassner. Interactive pop-up card design, part 1. *IEEE Computer Graphics and Applications*, 22(1):79–86, 2002.
- 13 Andrew Glassner. Interactive pop-up card design, part 2. *IEEE Computer Graphics and Applications*, 22(2):74–85, 2002.
- 14 Takuya Hara and Kokichi Sugihara. Computer-aided design of pop-up books with two-dimensional v-fold structures. In *Abstracts from the 7th Japan Conference on Computational Geometry and Graphs*, Kanazawa, Japan, November 2009.
- 15 Susan Hendrix. Popup workshop. <http://13d.cs.colorado.edu/~ctg/projects/popups/>, 2007.
- 16 Susan L. Hendrix and Michael A. Eisenberg. Computer-assisted pop-up design for children: computationally enriched paper engineering. *Advanced Technology for Learning*, 3(2), April 2006.
- 17 Peter Hilton and Jean Pedersen. Constructing pop-up polyhedra. In *Build Your Own Polyhedra*, chapter 7, pages 101–105. Addison-Wesley, 1994. Based on an article “Pop-up Polyhedra” by Jean Pedersen, *California Mathematics*, April 1983, pages 37–41.
- 18 Elliot E. Hui, Roger T. Howe, and M. Steven Rodgers. Single-step assembly of complex 3-D microstructures. In *Proceedings of the 13th Annual International Conference on Micro Electro Mechanical Systems*, pages 602–607, January 2000.
- 19 Satoshi Iizuka, Yuki Endo, Jun Mitani, Yoshihiro Kanamori, and Yukio Fukui. An interactive design system for pop-up cards with a physical simulation. *The Visual Computer*, 27(6):605–612, 2011. Proceedings of Computer Graphics International 2011.
- 20 Paul Jackson. *The Pop-Up Book: Step-by-Step Instructions for Creating Over 100 Original Paper Projects*. Holt Paperbacks, 1993.
- 21 Scott Johnson and Hans Walser. Pop-up polyhedra. *The Mathematical Gazette*, 81(492):364–380, 1997.
- 22 Xian-Ying Li, Chao-Hui Shen, Shi-Sheng Huang, Tao Ju, and Shi-Min Hu. Popup: Automatic paper architectures from 3D models. *ACM Transactions on Graphics*, 29(4):Article 111, 2010. Proceedings of SIGGRAPH 2010.
- 23 Jun Mitani and Hiromasa Suzuki. Computer aided design for origamic architecture models with polygonal representation. In *Proceedings of Computer Graphics International*, pages 93–99, 2004.
- 24 Jun Mitani, Hiromasa Suzuki, and Hiroshi Uno. Computer aided design for origamic architecture models with voxel data structure. *Transactions of Information Processing Society of Japan*, 44(5):1372–1379, 2003.
- 25 Sosuke Okamura and Takeo Igarashi. An assistant interface to design and produce a pop-up card. *International Journal of Creative Interfaces and Computer Graphics*, 1(2):40–50, 2010.
- 26 David Pelham. *Trail: Paper Poetry Pop-Up*. Little Simon, 2007.
- 27 Ellen G. K. Rubin. A history of pop-up and movable books: 700 years of paper engineering. Public lecture, November 10 2010. <http://www.youtube.com/watch?v=iDJJ0aZ1myM>.
- 28 I. Kh. Sabitov. On the problem of the invariance of the volume of a deformable polyhedron. *Uspekhi Mat. Nauk*, 50(2(302)):223–224, 1995.
- 29 Hugo Steinhaus. *Mathematical Snapshots*, pages 196–198. Oxford University Press, 1950. Republished by Dover Publications, 1999.
- 30 Tama Software. Pop-up card designer. [http://www.tamasoft.co.jp/craft/popupcard\\_en/](http://www.tamasoft.co.jp/craft/popupcard_en/), 2007. Pro version, [http://www.tamasoft.co.jp/craft/popupcard-pro\\_en/](http://www.tamasoft.co.jp/craft/popupcard-pro_en/), 2008.
- 31 Ryuhei Uehara and Sachio Teramoto. The complexity of a pop-up book. In *Proceedings of the 18th Annual Canadian Conference on Computational Geometry*, Ontario, Canada, August 2006.
- 32 Diego Uribe. *Fractal Cuts*. Tarquin, 1994.