

All Polyhedral Manifolds are Connected by a 2-Step Refolding

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Abstract:

We prove that, for any two polyhedral manifolds $\mathcal{P}, \mathcal{Q}$, there is a polyhedral manifold $\mathcal{I}$ such that $\mathcal{P}, \mathcal{I}$ share a common unfolding and $\mathcal{I}, \mathcal{Q}$ share a common unfolding. In other words, we can unfold $\mathcal{P}$, refold (glue) that unfolding into $\mathcal{I}$, unfold $\mathcal{I}$, and then refold into $\mathcal{Q}$. Furthermore, if $\mathcal{P}, \mathcal{Q}$ have no boundary and can be embedded in 3D (without self-intersection), then so does $\mathcal{I}$. These results generalize to n given manifolds $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$; they all have a common unfolding with the same intermediate manifold $\mathcal{I}$. Allowing more than two unfold/refold steps, we obtain stronger results for two special cases: for doubly covered convex planar polygons, we achieve that all intermediate polyhedra are planar; and for tree-shaped polycubes, we achieve that all intermediate polyhedra are tree-shaped polycubes.

Keywords: unfolding, refolding, polyhedra, manifolds, nonconvex

1. Introduction

Consider a *polyhedral manifold* — a connected two-dimensional surface made from flat polygons by gluing together paired portions of boundary (but possibly still leaving some boundary unpaired, and not necessarily embedded in space without overlap). Two basic operations on such a manifold are *gluing* (joining together two equal-length portions of remaining boundary) and the inverse operation *cutting* (splitting a curve into two equal-length portions of boundary, while preserving overall connectivity of the manifold). If we cut a manifold $\mathcal{P}$ enough that it can be laid isometrically into the plane (possibly with overlap), we call the resulting flat shape U an *unfolding* of $\mathcal{P}$.^{*1} Conversely, if we glue a flat shape U into any polyhedral manifold $\mathcal{P}$, we call $\mathcal{P}$ a *folding* of U (and U an unfolding of $\mathcal{P}$).

(Un)foldings naturally define an infinite bipartite graph $\mathbf{G}$ [1, Section 25.8.3]: define a vertex on one side for each man-

ifold $\mathcal{P}$, a vertex on the other side for each flat shape U , and an edge between U and $\mathcal{P}$ whenever U is an unfolding of $\mathcal{P}$ (or equivalently, $\mathcal{P}$ is a folding of U). Because unfolding and folding preserve surface area, we can naturally restrict the graph to manifolds and flat shapes of a fixed surface area A . Is the resulting graph $\mathbf{G}_A$ connected? In other words, is it possible to transform any polyhedral manifold into any other polyhedral manifold of the same surface area by an alternating sequence of unfolding to a flat shape, folding that flat shape into a new manifold, unfolding that manifold into a flat shape, and so on? We call each pair of steps — unfolding and then folding — a *refolding step*. We can then ask whether two manifolds have a k -step refolding for each $k = 1, 2, \dots$

In this paper, we give the first proof that the graph $\mathbf{G}_A$ is connected. In fact, we show that the graph has diameter at most 2: every two polyhedral manifolds $\mathcal{P}, \mathcal{Q}$ have a 2-step refolding. In other words, there is a single polyhedral manifold $\mathcal{I}$ such that $\mathcal{P}$ and $\mathcal{I}$ share a common unfolding, as do $\mathcal{I}$ and $\mathcal{Q}$. This result turns out to follow relatively easily using classic results from common dissection, similar in spirit to general algorithms for hinged dissection [4].

More interesting is that we show similar results when we restrict the polyhedral manifolds to the following special cases, which sometimes reduce the allowed input manifolds $\mathcal{P}, \mathcal{Q}$, but importantly also reduce the allowed intermediate manifolds $\mathcal{I}$:

(1) **No boundary:** If polyhedral manifolds $\mathcal{P}$ and $\mathcal{Q}$ have no boundary (what we might call “polyhedra”), then there is a 2-step refolding where the intermediate manifold $\mathcal{I}$ also has no boundary. This version is similarly

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^{*1} Note that our notion of “unfolding” differs from many other uses, such as [1], which forbid overlap. But it matches most previous work on refolding [2, 3].

easy. (In fact, we can achieve this property even when $\mathcal{P}$ and $\mathcal{Q}$ have boundary.)

- (2) **Embedded:** If polyhedral manifolds $\mathcal{P}$ and $\mathcal{Q}$ are embedded in 3D and have no boundary, then there is a 2-step refolding where the intermediate manifold $\mathcal{I}$ is embedded in 3D and has no boundary. This strengthening follows directly from a general theorem of Burago and Zalgaller [5]. (In fact, we can achieve this property whenever $\mathcal{P}$ and $\mathcal{Q}$ are orientable. Or, if we allow $\mathcal{I}$ to have boundary, we can always make it embeddable in 3D.)
- (3) **Doubly covered convex polygons:** If polyhedral manifolds $\mathcal{P}$ and $\mathcal{Q}$ are doubly covered convex polygons, then there is an $O(n)$ -step refolding where every intermediate manifold $\mathcal{I}$ is “planar” (all polygons lie in the plane, but possibly with multiple layers) and has no boundary. This result follows from an $O(1)$ -step refolding to remove a vertex from a doubly covered convex polygon.
- (4) **Polycubes:** If polyhedral manifolds $\mathcal{P}$ and $\mathcal{Q}$ are the surfaces of tree-shaped n -cubes (made from n unit cubes joined face-to-face according to a tree dual), then there is an $O(n^2)$ -step refolding where every intermediate manifold $\mathcal{I}$ is a (possibly self-intersecting) tree-shaped n -cube. Furthermore, the refoldings involve cuts only along edges of the cubes (*grid edges*); and if the given polycubes do not self-intersect and are “slit-free”, then the intermediate polycubes also do not self-intersect. This result follows from simulating operations in reconfigurable robots.

Past work on refolding has focused on the restriction to polyhedral manifolds that are the surfaces of *convex polyhedra*. This version began with a specific still-open question — is there a 1-step refolding from a cube to a regular tetrahedron? — independently posed by M. Demaine (1998), F. Hurtado (2000), and E. Pegg (2000). When E. Demaine and J. O’Rourke wrote this problem in their book [1, Open Problem 25.6], they also introduced the multi-step refolding problem. Let $\mathbf{C}_A$ be the subgraph of $\mathbf{G}_A$ restricting to convex polyhedra and their unfoldings. Demaine, Demaine, Diomidova, Kamata, Uehara, and Zhang [2] showed that several convex polyhedra of surface area A are all in the same connected component of $\mathbf{C}_A$: doubly covered triangles, doubly covered regular polygons, tetramonohedra (tetrahedra whose four faces are congruent acute triangles, including doubly covered rectangles), regular prisms, regular prismatoids, augmented regular prismatoids, and all five Platonic solids. These refoldings require just $O(1)$ steps (at most 9).

Our 2-step refolding is very general, applying in particular to any two convex polyhedra; for example, Figure 1 shows the example of a cube to a regular tetrahedron. But our refolding crucially relies on a nonconvex intermediate manifold $\mathcal{I}$. We conjecture that two steps is also optimal, even for two convex polyhedra. Indeed, Arseneva, Demaine, Kamata, and Uehara [3] conjectured that most pairs of doubly covered triangles (specifically, those with rationally indepen-

dent angles) have no common unfolding, and thus no 1-step refolding. As evidence, they showed (by exhaustive search) that any common unfolding has at least 300 vertices. Assuming this conjecture, two steps are sometimes necessary. Two steps is also the first situation where we have an intermediate manifold $\mathcal{I}$, which is what allows us to exploit the additional freedom of the nonconvexity of $\mathcal{I}$.

After presenting our general 2-step refolding (Section 3), we consider the special cases of doubly covered convex polygons (Section 4) and polycubes (Section 5).

2. Refolding Model

In our constructions, we use a more general but equivalent form of “refolding step”: any cutting followed by any gluing. In other words, we allow using an arbitrary connected manifold in between cutting and gluing. By contrast, the definition in Section 1 requires a full unfolding followed by a folding, which requires a flat shape in between cutting and gluing.

These two models are equivalent. If we want to modify one of our refolding steps to instead reach a full unfolding after cutting, we can perform additional cuts (that preserve connectivity) until the manifold can be laid flat, and then immediately reglue those cuts back together. The same idea is used in [2].

3. Transformation Between Polyhedral Manifolds

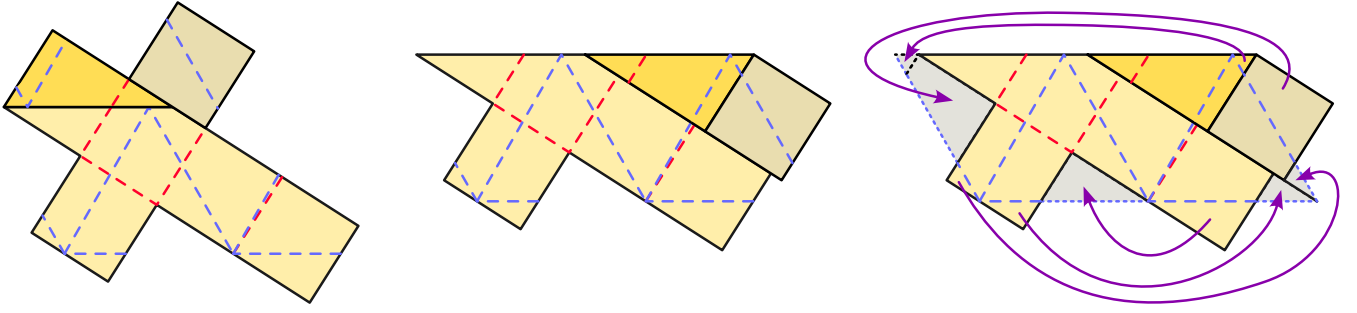
In this section, we prove the main result of the paper:

Theorem 3.1. *For any n polyhedral manifolds $\mathcal{P}_1, \dots, \mathcal{P}_n$ of the same surface area, there is another polyhedral manifold $\mathcal{I}$ such that $\mathcal{P}_i$ and $\mathcal{I}$ have a common unfolding for all i . We can guarantee that the intermediate manifold $\mathcal{I}$ has no boundary, or guarantee that it embeds in 3D. If manifolds $\mathcal{P}_i$ are all orientable, then we can guarantee that the intermediate manifold $\mathcal{I}$ is orientable, has no boundary, and embeds in 3D.*

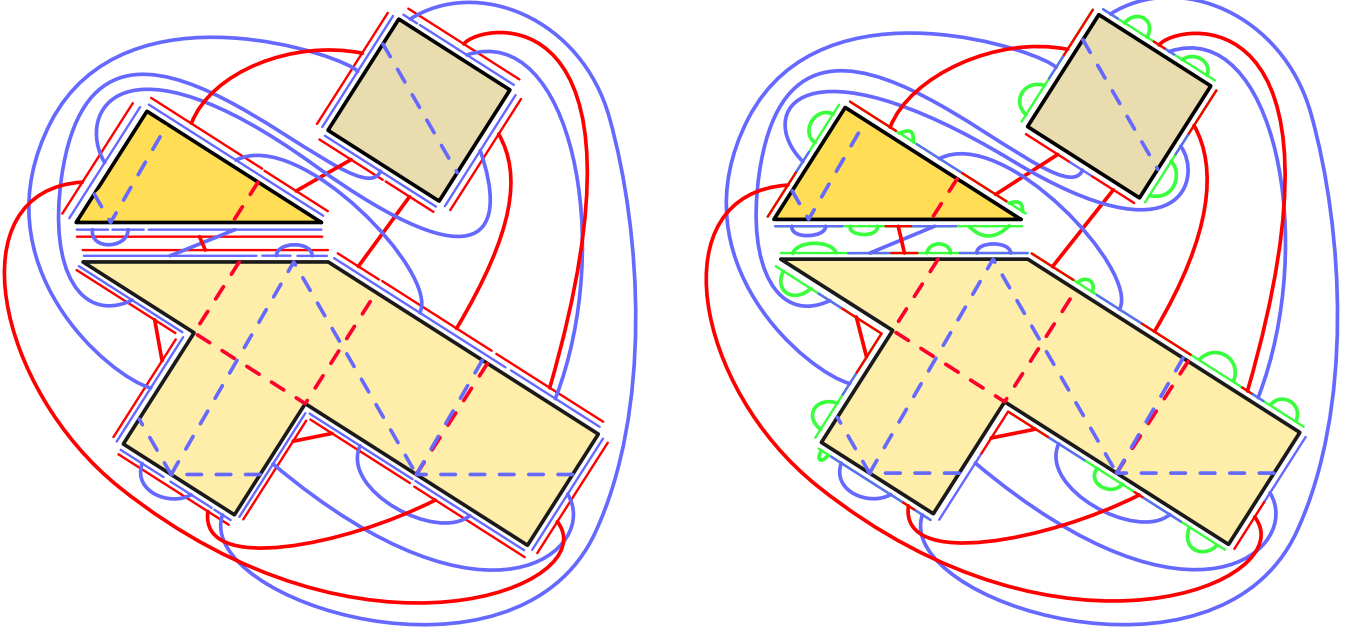
We initially focus on the case of $n = 2$ manifolds:

Corollary 3.2. *Any two polyhedral manifolds $\mathcal{P}, \mathcal{Q}$ of the same surface area have a 2-step refolding. We can guarantee that the intermediate manifold $\mathcal{I}$ has no boundary, or guarantee that it embeds in 3D. If manifolds $\mathcal{P}$ and $\mathcal{Q}$ are both orientable, then we can guarantee that the intermediate manifold $\mathcal{I}$ is orientable, has no boundary, and embeds in 3D.*

Figure 1 gives an example construction of an intermediate manifold $\mathcal{I}$ when $\mathcal{P}$ is a cube and $\mathcal{Q}$ is a regular tetrahedron. To improve figure clarity, this construction does not exactly follow the general algorithm described below: instead of the general dissection algorithm, we use an efficient 3-piece dissection based on a dissection of Gavin Theobald [6], and we preserve more original gluings when resolving overlaps. Nonetheless, it serves as a running example of the key steps in our algorithm.



(a) 3-piece dissection of the cube into the regular tetrahedron, based on Gavin Theobald's 5-piece dissection of the Latin cross into the equilateral triangle [6]. Red and blue dashed lines represent the folding to a cube and regular tetrahedron respectively.



(b) Desired gluings for the cube (red) and the regular tetrahedron (blue).

(c) Gluing for an intermediate manifold $\mathcal{I}$, from subsets of the cube (red) and the tetrahedron (blue) gluing and zipping the remainder (green).

Fig. 1: Example 2-step refolding from the cube to the regular tetrahedron.

3.1 Common Dissection

We start by computing a *common dissection* of the given manifolds $\mathcal{P}$ and $\mathcal{Q}$ of equal surface area, that is, a subdivision of each surface into polygons that match in the sense that, for some perfect pairing of $\mathcal{P}$'s polygons with $\mathcal{Q}$'s polygons, there is an isometry between paired polygons. Solutions to this dissection problem for *polygons* $\mathcal{P}$ and $\mathcal{Q}$ go back to the early 1800s [7, 8, 9, 10]. Their high-level approach is as follows:

- (1) Triangulate $\mathcal{P}$ and $\mathcal{Q}$.
- (2) Dissect each triangle (from both triangulations) into a rectangle. (This dissection needs only three pieces: cut the triangle parallel to its base at half the height, and cut from the apex orthogonal to the first cut.)
- (3) Dissect each rectangle into a rectangle of height $h_{\min}$, the smallest height among all the rectangles (i.e., half the smallest height among all triangles in both triangulations). (This dissection is more difficult and can require a polynomial number of **[Someone fact-check this —ALICE]** pieces in the general case.)

- (4) Arrange all the rectangles from $\mathcal{P}$ into one long rectangle of height $h_{\min}$, and similarly arrange all the rectangles from $\mathcal{Q}$ into one long rectangle of height $h_{\min}$, necessarily the same rectangle R .
 - (5) Overlay the two dissections of this common rectangle R and subdivide according to all cuts, producing a set of polygons that can form into $\mathcal{P}$ and can form into $\mathcal{Q}$.
- See [4, 11] for more algorithmic descriptions, including pseudopolynomial bounds on the number of pieces.

We can apply the same technique to the case where $\mathcal{P}$ and $\mathcal{Q}$ are polyhedral manifolds instead of polygons. The only slightly different step is triangulating the surfaces $\mathcal{P}$ and $\mathcal{Q}$ (Step 1), which we can do by e.g. triangulating the faces. Then Steps 2–5 apply to the resulting triangles as usual. Because the dissection construction does not require flipping the polygons, the resulting common dissection is *locally orientation preserving*: the mapping from the polygons arranged to form $\mathcal{P}$ to the polygons arranged to form $\mathcal{Q}$ locally preserves which side of each polygon is “up”.

The resulting dissection may not be “edge-to-edge”: when

assembling the polygons together to form $\mathcal{P}$ or $\mathcal{Q}$, two polygons may meet (intersect) at a segment that is only a subset of an edge of either polygon. Figure 1a shows an example of such a dissection, from a cube to a regular tetrahedron: for example, in the cube (cross) arrangement, the triangular piece and square piece share only a portion of their edges. (This example was designed by hand, based on a dissection by Gavin Theobald [6], not produced by the algorithm above. In fact, Gavin Theobald found a 2-piece dissection from the cube to the regular tetrahedron [12].)

We can generalize this common dissection construction to n polyhedral manifolds $\mathcal{P}_1, \dots, \mathcal{P}_n$ of the same surface area (as also mentioned in [4]): just overlay n dissections in Step 5. Henceforth we will consider the case of general n .

3.2 Abstract Intermediate Manifold

Next we construct the intermediate manifold $\mathcal{I}$. For now, we will not worry about embeddability, and just construct an (abstract) polyhedral manifold.

Consider two polygons P_1, P_2 in the common dissection that are adjacent in manifold $\mathcal{P}_i$ meaning that, when the polygons are assembled to form $\mathcal{P}_i$, there is an edge e_1 of P_1 and an edge e_2 of P_2 that overlap on a common positive-length segment. Let $\cap_i(e_1, e_2)$ denote the segment of e_1 that intersects e_2 when assembling $\mathcal{P}_i$, and let $\cap_i(e_2, e_1)$ denote the corresponding segment of e_2 . (If the edges share more than one segment, as they might in a non-edge-to-edge gluing, pick one arbitrarily, but consistently for e_1 and e_2 .) Intuitively, $\cap_i(e_1, e_2) \leftrightarrow \cap_i(e_2, e_1)$ represent the gluings desired by $\mathcal{P}_i$, but the $\mathcal{P}_i$ gluings likely conflict with the $\mathcal{P}_j$ gluings for $i \neq j$. Figure 1b gives an example.

Next we construct a (partial) gluing on the boundaries of the polygons that includes a positive segment from every $\cap_i(e_1, e_2) \leftrightarrow \cap_i(e_2, e_1)$ gluing, while avoiding conflicts. The algorithm proceeds as follows:

- (1) For each manifold $\mathcal{P}_i$, and for every overlapping pair of edges e_1 of P_1 and e_2 of P_2 when assembling the polygons into $\mathcal{P}_i$, add $\cap_i(e_1, e_2) \leftrightarrow \cap_i(e_2, e_1)$ to the list of gluings.
- (2) Find two gluings that overlap on an edge e , say $s_1 \leftrightarrow s_2$ and $s'_1 \leftrightarrow s'_2$ where segments s_1 and s'_1 are subsegments of a common edge e that overlap on a positive-length segment $s_1 \cap s'_1$. Refer to Figure 2.

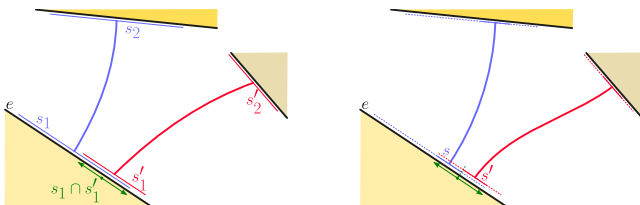


Fig. 2: Removing overlap from two gluings $s_1 \leftrightarrow s_2$ and $s'_1 \leftrightarrow s'_2$ on edge e , by reducing s_1 and s'_1 to subsegments s and s' which bisect $s_1 \cap s'_1$.

- (3) Divide the segment $s_1 \cap s'_1$ into two equal halves, s and s' .

- (4) Remove the overlap between these two gluings by restricting $s_1 \leftrightarrow s_2$ to the subsegment $s \subset s_1$, and restricting $s'_1 \leftrightarrow s'_2$ to the subsegment $s' \subset s'_1$.
- (5) Repeat Steps 2–4 until all overlaps have been removed. Because this algorithm only modifies gluings by restricting to a subsegment, it never adds new overlaps, so it will remove all overlaps after $O((nE^2)^2)$ repetitions, where E is the number of edges in the common dissection so $O(E^2)$ bounds the number of gluings from each manifold $\mathcal{P}_i$. Furthermore, every original gluing $\cap_i(e_1, e_2) \leftrightarrow \cap_i(e_2, e_1)$ remains intact for some positive length.

Some of the boundary of the polygons may now be unglued. If we want to avoid $\mathcal{I}$ having boundary, we can glue each segment s of remaining boundary to itself, by dividing it in two equal halves s', s'' , giving s' and s'' opposite orientations, and gluing $s' \leftrightarrow s''$. (This type of gluing is called “zipping” [13].) Figure 1c gives an example of a gluing that might result from an optimized form of this algorithm (where we maintain as much of the original gluings as possible).

The gluing described above defines the intermediate manifold $\mathcal{I}$. Because manifold $\mathcal{I}$ contains a portion of every desired gluing for $\mathcal{P}_i$, $\mathcal{I}$ has a common unfolding with $\mathcal{P}_i$: just cut all gluings that did not originate from $\mathcal{P}_i$. Because no cut fully separates an entire edge from its mate in $\mathcal{P}_i$, this cutting preserves connectivity; indeed, we obtain the same dual graph of piece adjacencies as we do when arranging the dissection into $\mathcal{P}_i$. Thus we obtain a 1-step refolding from $\mathcal{P}_i$ to $\mathcal{I}$, and similarly from $\mathcal{I}$ to $\mathcal{P}_j$, which gives a 2-step refolding from $\mathcal{P}_i$ to $\mathcal{P}_j$.

If manifolds $\mathcal{P}_i$ are all orientable, then so is the resulting intermediate manifold $\mathcal{I}$, because the common dissection is (locally) orientation preserving.

3.3 Embeddable Intermediate Polyhedron via Burago–Zalgaller Theorem

To guarantee that $\mathcal{I}$ is embeddable in 3D, we use a powerful result of Burago and Zalgaller [5]. See also [14] for a detailed description of the result, and [15] for a description of the (quite complicated) construction.

Theorem 3.3 ([5, Theorem 1.7]). *Every polyhedral manifold that is either orientable or has boundary admits an isometric piecewise-linear C^0 embedding into 3D.*

To apply this theorem, we need that $\mathcal{I}$ is either orientable or has boundary. As argued above, $\mathcal{I}$ is orientable if manifolds $\mathcal{P}_i$ are all orientable. Otherwise, we can give $\mathcal{I}$ boundary by reducing any one gluing to half of its length (or omitting the zips if we had some). Either way, Theorem 3.3 gives a subdivision of the polygons in $\mathcal{I}$ into finitely many subpolygons, each of which gets isometrically embedded in 3D by an embedding of $\mathcal{I}$.

4. Transformation Between Doubly Covered Convex Polygons

We start with *doubly covered convex polygons*, that is, polyhedral manifolds without boundary formed from two

copies of a convex planar polygon by gluing together all corresponding pairs of edges. Here we require that every intermediate polyhedral manifold $\mathcal{I}$ is **planar** in the sense that its polygons all lie in the plane, but we allow any number of layers of stacked polygons, generalizing the notion of doubly covered polygon.

Theorem 4.1. *Any two doubly covered convex n -gons of the same area have an $O(n)$ -step refolding, where all intermediate manifolds are planar with no boundary.*

As a useful building block, we consider a simple 2-step refolding which allows removing a piece from the polygon, rotating it, and gluing it back elsewhere (similar to hinged dissection), provided we can fold the polygon to facilitate the gluing.

Lemma 4.2. *Let P be a subset of the plane homeomorphic to a closed disk, as visualized in Figure 3. Suppose A_1B_1 and A_2B_2 are line segments on the boundary of P , such that there exists a plane reflection r taking A_1 to A_2 and B_1 to B_2 . Let c be a simple curve starting and ending on the boundary of P passing through its interior, so that it separates P into two closed halves P_1 and P_2 containing A_1B_1 and A_2B_2 respectively. Let f be the unique rotation and translation such that $f(A_1) = A_2$ and $f(B_1) = B_2$, and suppose $f(P_1)$ intersects P_2 only on A_2B_2 . Then there is a 2-step refolding between the double covers of P and P' , where $P' = f(P_1) \cup P_2$.*

Proof. The refolding is accomplished by the following steps, illustrated in Figure 3:

- (1) Cut along A_1B_1 and A_2B_2 , then glue the top layer of A_1B_1 to the top layer of A_2B_2 and similarly for the bottom layers. This intermediate step can be folded flat by a single fold along the line of reflection of r .
- (2) Cut along c in both layers to create two new boundaries c_1 and c_2 . Then glue the top layer of c_1 to the bottom layer of c_1 and similarly for c_2 . $\square$

Now we consider the triangle $\triangle ABC$ formed by three consecutive vertices A, B, C on the boundary of a polygon P . Our goal is to find an $O(1)$ -step refolding of P which moves the apex B parallel to AC (which preserves area). This will allow us to move B so that the interior angle at C becomes 180° , eliminating a vertex from P . By induction, this allows us to reduce any doubly covered polygon down to a doubly covered triangle, and then we can use a known 3-step refolding between doubly covered triangles [2, Theorem 2]. We accomplish the goal as follows:

Lemma 4.3. *Let P be a convex polygon with three consecutive vertices A, B, C such that the projection of B onto AC is between A and C . Then there is an $O(1)$ -step refolding between the double covers of P and P' , where P' is the polygon obtained from P by replacing $\triangle ABC$ by a rectangle with base AC with the same area.*

Proof. Refer to Figure 4. Let X be the midpoint of AB , Y be the midpoint of BC , and O be the projection of B onto XY . Using Lemma 4.2, we rotate $\triangle XBO$ by 180° about

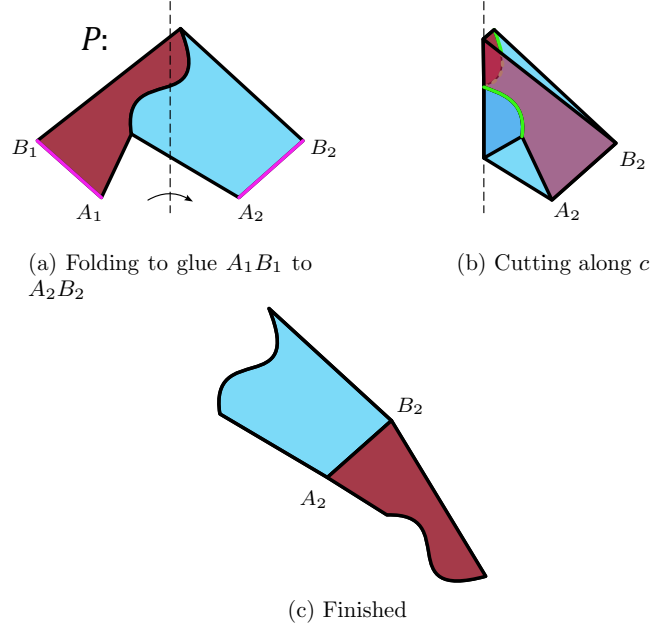


Fig. 3: Rearranging two pieces via a 2-step refolding.

X , and similarly we rotate $\triangle YBO$ by 180° about Y . This forms the desired rectangle. $\square$

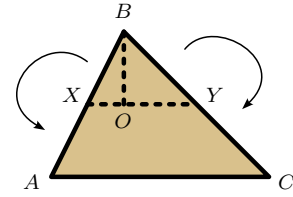


Fig. 4: Refolding $\triangle ABC$ into a rectangle when B is between A and C .

Lemma 4.4. *Let P be a convex polygon with three consecutive vertices A, B, C , and let B_1 be the projection of B onto AC . Suppose that C is between A and B_1 , and $|CB_1| \leq 4|AC|$. Then there is an $O(1)$ -step refolding between the double covers of P and P' , where P' is the polygon obtained from P by replacing $\triangle ABC$ by a rectangle with base AC with the same area.*

Proof. Refer to Figure 5, where all point labels remain fixed in the plane across all subfigures. Let X be the midpoint of AB and Y be the midpoint of BC (Figure 5a). For this proof, we will adopt the convention that p_1 denotes the projection of point p onto AC and p_2 denotes the projection of p onto XY .

Using Lemma 4.2, we rotate $\triangle XYB$ by 180° about X (Figure 5a), forming the parallelogram $ACY Y'$ (Figure 5b). Now let W be the midpoint of CY and V be the midpoint of AY' . We have

$$|AV_1| = |W_2Y| = \frac{1}{4}|CB_1| \leq |AC| = |Y'Y|,$$

which implies V_1 lies on AC and W_2 lies on $Y'Y$. Using Lemma 4.2 twice, we rotate $\triangle AVV_1$ by 180° about V , and $\triangle YWW_2$ by 180° about W . This forms a rectangle $V_1V_2W_2W_1$. Finally, we use Lemma 4.2 again to move the

rectangle $CC_2W_2W_1$ to $AA_2V_1V_2$.

□

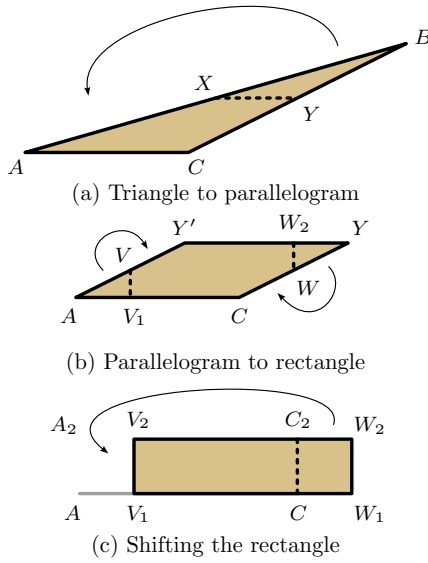


Fig. 5: Refolding $\triangle ABC$ into a rectangle when B is not between A and C .

Lemma 4.5. *Let P be a convex polygon with three consecutive vertices A, B, C and let Z be another vertex of P such that the interior angle at Z is at most the interior angle at B . Let ℓ be the line through B parallel to AC , Q_1 be the intersection of ZA with ℓ , and Q_2 be the intersection of ZC with ℓ . Then $\min\{|Q_1B|, |Q_2B|\} \leq |AC|$.*

Proof. Refer to Figure 6. Construct B' so that $ABCB'$ is a parallelogram. Vertex Z cannot lie in the interior of $\triangle ACB'$ or else its interior angle would be larger than that of B (by convexity of P). Thus Z is either below AB' or below CB' ; in the first case, we have $|Q_1B| \leq |AC|$, and in the second case, we have $|Q_2B| \leq |AC|$. □

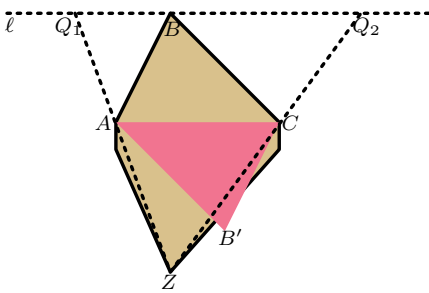


Fig. 6: Bounding the distance from B to Q_1, Q_2 .

Corollary 4.6. *Let P be a convex polygon with five consecutive vertices D_1, A, B, C, D_2 (where possibly $D_1 = D_2$) such that the interior angle at B is at least as large as the interior angles at D_1 and D_2 . Let ℓ be the line through B parallel to AC , Q_1 be the intersection of D_1A with ℓ , and Q_2 be the intersection of D_2C with ℓ . Then $\min\{|Q_1B|, |Q_2B|\} \leq |AC|$.*

Proof. By convexity of P , D_1 is below D_2A and so $|Q_1B|$ is at most the distance from B to the intersection of D_2A

with ℓ . The inequality thus follows from Lemma 4.5 applied to D_2 . □

Proof of Theorem 4.1. It suffices to show that, for $n \geq 4$, any doubly covered convex n -gon can be reduced to a doubly covered convex $(n-1)$ -gon by an $O(1)$ -step refolding, because then we can reduce both polygons to triangles in $O(n)$ steps, and [2, Theorem 2] shows there is a 3-step refolding between any pair of doubly covered triangles with the same area.

Let P be a convex n -gon where $n \geq 4$, and let B be a vertex of P with the largest interior angle. By Corollary 4.6 we can label the nearby vertices of B by A, C, D such that A, B, C, D are consecutive and $|QB| \leq |AC|$ where Q is the intersection of DC with ℓ , the line through B parallel to AC . Let P' be the polygon obtained from P by replacing $\triangle ABC$ by a rectangle with base AC of the same area, and let P'' be the polygon obtained from P by replacing $\triangle ABC$ by $\triangle AQC$. By Lemma 4.3, there is an $O(1)$ -step refolding between the double covers of P and P' ; it applies because the interior angle of B is at least 90° . Similarly, one of Lemmas 4.3 or 4.4 (using $|QB| \leq |AC|$) shows that there is an $O(1)$ -step refolding between the double covers of P'' and P' . Thus there is an $O(1)$ -step refolding between double covers of P and P'' . But because D, C, Q are collinear, P'' is a convex $(n-1)$ -vertex polygon. □

5. Transformation Between Tree-Shaped Polycubes

Next we consider *tree-shaped n -cubes*, that is, polyhedral manifolds formed from n unit cubes in 3D *joined* face-to-face in a tree structure (forming a tree dual graph). Here, when two cubes get joined together at a common face, we remove that face from the manifold, preserving that the manifold is homeomorphic to a sphere. (This notion of “join” is a higher-dimensional analog of gluing.) Thus every tree-shaped n -cube has surface area $6n - 2(n-1) = 4n + 2$.

We allow two cubes to be adjacent even if they are not glued together, in which case there are two surface squares in between. If there are no such touching cubes, we call the tree-shaped n -cube *slit-free*.^{*2} When the n -cubes are not slit-free, we further allow multiple cubes to occupy the same location in space, in which case we call the tree-shaped n -cube *self-intersecting*.

All cubes of a tree-shaped n -cube naturally lie on a cubical grid. Define *grid cutting* to be cutting restricted to edges of the cubical grid, and *grid refolding* to be grid cutting followed by gluing that results in another tree-shaped n -cube.

Theorem 5.1. *Any two tree-shaped n -cubes have an $O(n^2)$ -step grid refolding, where all intermediate manifolds are possibly self-intersecting tree-shaped n -cubes. If the*

^{*2} Our definition of “slit-free” here is less restrictive than previous notions of “well-separated” [16], which required at least one straight cube (connected to cubes on two opposite faces and nowhere else) between every two non-straight cubes.

given tree-shaped n -cubes do not self-intersect and are slit-free, then the intermediate manifolds do not self-intersect.

To transform between two given tree-shaped polycubes $\mathcal{P}$ and $\mathcal{Q}$, we mimic the “sliding cubes” model of reconfiguring modular robots made up of n cubes, which was recently solved in optimal $O(n^2)$ steps [17]. This model defines two types of operations (see Figure 7):

- (1) **Slide** a cube along a flat surface of neighboring cubes by 1 unit.
- (2) **Rotate** a cube around the edge of an adjacent cube.

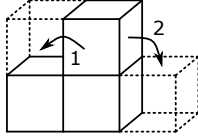


Fig. 7: Two different ways an individual cube can move on a surface of a polycube: (1) sliding and (2) rotating.

We will show how to perform slide and rotate operations for a **leaf** cube, that is, a leaf of the dual tree in a tree-shaped polycube. In this case, sliding can be viewed as moving a leaf cube to a new parent, and rotating can be viewed as the leaf cube attaching to a different location of the same parent.

To slide a leaf cube, we perform the following refolding step, illustrated in Figure 8:

- (1) Cut AB , BE , ED , FG , GJ , and IJ . These cuts free up the leaf cube to move into the adjacent location, as drawn in the intermediary figure in Figure 8.
- (2) Glue AB' to $E'B'$, FG' to $J'G'$, $E'D$ to AB , $J'I$ to FG , DE to BE , and IJ to GJ .

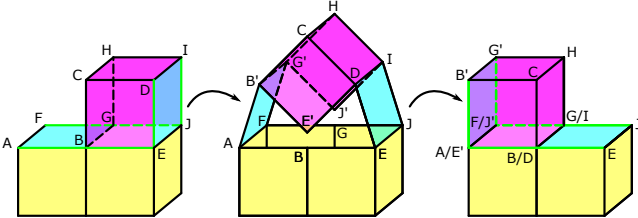


Fig. 8: Sliding a leaf cube.

Figure 9 shows an extension of sliding. Here the leaf cube $IDEJCHGB$ does not move, but it changes its parent from the cube attached below to the cube attached on its left, effectively traversing the reflex corner. The same refolding step as sliding applies in this case.

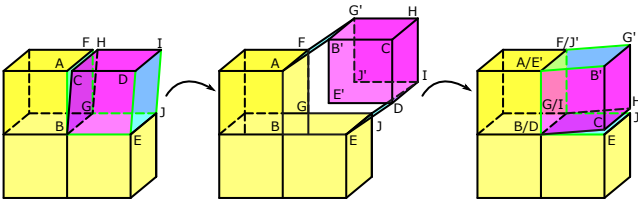


Fig. 9: “Sliding” a leaf cube in a reflex corner.

To rotate a leaf cube around an edge we perform the following refolding step, illustrated in Figure 10:

- (1) Cut BA , AD , DE , GF , FI , and IJ . Similar to the sliding procedure, these cuts free up the leaf cube to move, as shown in the intermediary figure in Figure 10.
- (2) Glue AB to AD , FG to FI , BA' to DE , GF' to IJ , $A'D'$ to ED' , and $F'I'$ to $J'I'$.

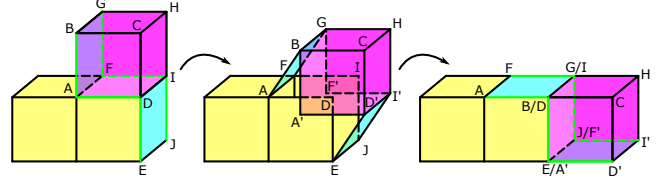


Fig. 10: Rotating a leaf cube over the edge of a polycube.

By combining these three operations, we can follow a simple algorithm for transforming a given n -cube $\mathcal{P}$ into a $1 \times 1 \times n$ line:

- (1) Fix one leaf cube as the **root** cube c_1 . Assume by symmetry that the root cube’s unique neighbor is in the down direction.
- (2) For $i = 2, 3, \dots, n$:
 - (a) Assume $c_1, \dots, c_{i-1}$ have been arranged into an upward line, with c_i being a current leaf.
 - (b) Take a leaf cube c_i that is not c_{i-1} (given that there are always at least two leaves).
 - (c) Slide and rotate c_i around the boundary of the rest of the tree until it reaches the root cube c_1 , and then slide it up the line to place it immediately above c_{i-1} .

This algorithm requires $O(n^2)$ steps: potentially each of the n cubes needs to traverse the surface area of the tree-shaped n -cube, which is $O(n)$. It may also cause self-intersection, because it blindly follows the surface of the tree-shaped n -cube, so it may place the moving leaf cube on top of an adjacent cube in the case of touching cubes. If the tree-shaped n -cube is slit-free, though, then this simple algorithm avoids self-intersection.

To transform between two tree-shaped n -cubes $\mathcal{P}$ and $\mathcal{Q}$, we apply the algorithm above separately to each of $\mathcal{P}$ and $\mathcal{Q}$, perform the refolding steps on $\mathcal{P}$ to transform it into a line, and then perform the reverse refolding steps on $\mathcal{Q}$ to transform the line into $\mathcal{Q}$. (Note that each refolding step is reversible.) Thus we have proved Theorem 5.1.

It is tempting to apply the (much more complicated) $O(n^2)$ -step algorithm of Abel, Akitaya, Kominers, Korman, and Stock [17], which has the advantage of avoiding self-intersection without any assumption of slit-freeness. Unfortunately, sliding and rotating nonleaf cubes seem more difficult. One approach is to transform one spanning tree into another (probably increasing the number of steps), but it is not even clear whether this can be accomplished by leaf reparenting operations.

It also seems likely that some of these moves can be done in parallel in the same refolding step, leading to fewer refolding steps. Some models of modular robotics have parallel reconfiguration algorithms that move a linear number of

robots in each round [18, 19]. It remains open whether we can get similarly good bounds in the cube sliding model or the leaf-focused sliding-by-refolding model.

6. Conclusion

In this paper, we showed a transformation algorithm between any two manifolds with two cut-and-glue refolding steps. When transforming between manifold $\mathcal{P}$ and manifold $\mathcal{Q}$, we go through an intermediate embeddable polyhedron which is not necessarily convex. We also showed two simpler refolding algorithms for doubly covered polyhedra and tree-shaped polycubes.

Many open questions remain:

- Are there examples where 1-step refolding is impossible? [3]
- If the two given polyhedra are convex, is there a finite-step refolding where the intermediate polyhedra are also convex? [1, Section 25.8.3]
- Can we extend our polycube result to avoid self-intersection without assuming slit-freeness, or to support non-tree-shaped polycubes of the same surface area?
- Can we improve the number of refolding steps needed for the doubly covered polygon or polycube refolding algorithms?

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