Any Platonic Solid Can Transform to Another by $O(1)$ Refoldings

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Abstract

We show that several classes of polyhedra are joined by a sequence of $O(1)$ refolding steps, where each refolding step unfolds the current polyhedron (allowing cuts anywhere on the surface and allowing overlap) and folds that unfolding into exactly the next polyhedron; in other words, a polyhedron is refoldable into another polyhedron if they share a common unfolding. Specifically, assuming equal surface area, we prove that (1) any two tetramonohedra are refoldable to each other, (2) any doubly covered triangle is refoldable to a tetramonohedron, (3) any (augmented) regular prismatoid and doubly covered regular polygon is refoldable to a tetramonohedron, (4) any tetrahedron has a 3-step refolding sequence to a tetramonohedron, and (5) the regular dodecahedron has a 4-step refolding sequence to a tetramonohedron. In particular, we obtain a ≤ 6-step refolding sequence between any pair of Platonic solids, applying (5) for the dodecahedron and (1) and/or (2) for all other Platonic solids. As far as the authors know, this is the first result about common unfolding involving the regular dodecahedron.

Keywords: Unfolding of Polyhedra, Common Unfolding, Refolding Dissection, Reconfiguration Problem

1. Introduction

A polyhedron Q is ***refoldable*** to a polyhedron Q' if Q can be unfolded to a planar shape that folds into exactly the surface of Q', i.e., Q and Q' share a common unfolding/development, allowing cuts anywhere on the surfaces of Q and Q'. (Although it is probably not necessary for our refoldings, we also allow the common unfolding to self-overlap, as in [8].) The idea of refolding was proposed independently by M. Demaine, F. Hurtado, and E. Pegg [1, Open Problem 25.6], who specifically asked whether every regular polyhedron (Platonic solid) can be refolded into any other regular polyhedron. In this context, there exist some specific results: Araki et al. [9] found two Johnson-Zalgaller solids that are refoldable to regular tetrahedra [9], and Shirakawa et al. [3] found an infinite sequence of polygons that can each fold into a cube and an approaching-regular tetrahedron.

More broadly, Demaine et al. [6] showed that any convex polyhedron can always be refolded to at least one other convex polyhedron. Xu et al. [2] and Biswas and Demaine [10] found common unfoldings of more than two (specific) polyhedra. On the negative side, Horiyama and Uehara [8] proved impossibility of certain refoldings when the common unfolding is restricted to cut along the edges of polyhedra.

In this paper, we consider the connectivity of polyhedra by the transitive closure of refolding, an idea suggested by Demaine and O'Rourke [1, Section 25.8.3]. Define a ***(k-step) refolding sequence*** from Q to Q' to be a sequence of convex polyhedra $Q = Q_0, Q_1, \ldots, Q_k = Q'$ where each Q_{i-1} is refoldable to Q_i. We refer to k as the ***length*** of the refolding sequence. We just say "refoldable" when two polyhedra have a 1-step refolding sequence.

Our results. Do all pairs of convex polyhedra of the same surface area (a trivial necessary condition) have a finite-step refolding sequence? If so, how short of a sequence suffices? As mentioned in [1, Section 25.8.3], the regular polyhedron open problem mentioned above is equivalent to asking whether 1-step refolding sequences exist for all pairs of regular polyhedra. We solve a closely related problem, replacing "1" with "$O(1)$": for any pair of regular polyhedra Q and Q', we give a refolding sequence of length at most 6.

More generally, we give a series of results about $O(1)$-step refolding certain pairs of polyhedra of the same surface area:

1. In Section 3, we show that any two tetramonohedra are refoldable to each other, where a ***tetramonohedron*** is a tetrahedron that consists of four congruent acute triangles.
   This result offers a possible "canonical form" for finite-step refolding sequences between any two polyhedra: because a refolding from Q to Q' is also a refolding from Q' to Q, it suffices to show that any polyhedron has a finite-step refolding into some tetramonohedron.
2. In Section 4, we show that every regular prismatoid and every augmented regular prismatoid are refoldable to a tetramonohedron.
   In particular, the regular tetrahedron is a tetramonohedron, the regular hexahedron (cube) is a regular prismatoid, and the regular octahedron and regular icosahedron are both augmented regular prismatoids. Therefore, the regular tetrahedron has a 2-step refolding sequence to the regular hexahedron, octahedron, and icosahedron (via an intermediate tetramonohedron); and every pair of polyhedra among the regular hexahedron, octahedron, and icosahedron have a 3-step refolding sequence (via two intermediate tetramonohedra).
3. In Section 5, we prove that a regular dodecahedron is refoldable to a tetramonohedron by a 4-step refolding sequence.
   As far as the authors know, there are no previous explicit refolding results for the regular dodecahedron, except the general results of [6].
   Combining the results above, any pair of regular polyhedra (Platonic solids) have a refolding sequence of length at most 6.

4. In addition, we prove that every doubly covered triangle (Section 3) and every doubly covered regular polygon (Section 4) are refoldable to a tetramonohedron, and that every tetrahedron has a 3-step refolding sequence to a tetramonohedron (Section 6).
   Therefore, every pair of polyhedra among the list above have an $O(1)$-step refolding sequence.

2. Preliminaries

For a polyhedron Q, $V(Q)$ denotes the set of vertices of Q. For $v \in V(Q)$, define the ***cocurvature*** $\sigma(v)$ of v on Q to be the sum of the angles incident to v on the facets of Q. The ***curvature*** $\kappa(v)$ of v is defined by $\kappa(v) = 2\pi - \sigma(v)$. In particular, if $\kappa(v) = \sigma(v) = \pi$, we call v a ***smooth vertex***. We define $\boldsymbol{\Pi_k}$ to be the class of polyhedra Q with exactly k smooth vertices. It is well-known that the total curvature of the vertices of any convex polyhedron is 4π, by the Gauss–Bonnet Theorem (see [1, Section 21.3]). Thus the number of smooth vertices of a convex polyhedron is at most 4. Therefore, the classes $\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ give us a partition of all convex polyhedra.

An ***unfolding*** of a polyhedron is a (possibly self-overlapping) planar polygon obtained by cutting and developing the surface of the polyhedron (allowing cuts anywhere on the surface). ***Folding*** a polygon P is an operation to obtain a polyhedron Q by choosing crease lines on P and gluing the boundary of P properly. When the polyhedron Q is convex, the following result is crucial:

Lemma 1 (Alexandrov's Theorem [12, 1]). *If we fold a polygon P in a way that satisfies the following three* ***Alexandrov's conditions****, then there is a unique convex polyhedron Q realized by the folding.*

1. *Every point on the boundary of P is used in the gluing.*
2. *At any glued point, the summation of interior angles (cocurvature) is at most 2π.*
3. *The obtained surface is homeomorphic to a sphere.*

By this result, when we fold a polygon P to a convex polyhedron Q, it is enough to check that the gluing satisfies Alexandrov's conditions. (In this paper, it is easy to check that the conditions are satisfied by our (re)foldings, so we omit their proof.)

A polyhedron Q is ***(1-step) refoldable*** to a polyhedron Q' if Q can be unfolded to a connected polygon that folds to Q' (and thus they have the same surface area). A ***(k-step) refolding sequence*** of a polyhedron Q to a polyhedron Q' is a sequence of convex polyhedra $Q = Q_0, Q_1, \ldots, Q_k = Q'$ where Q_{i-1} is refoldable to Q_i for each $i \in \{1, \ldots, k\}$. To simplify some arguments that Q is refoldable to Q', we sometimes only partially unfold Q (cutting less than needed to make the surface unfold flat), and refold to Q' so that Alexandrov's conditions hold.

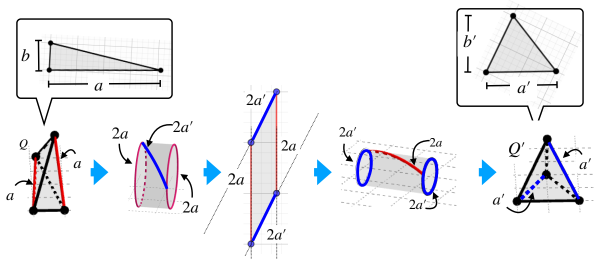

Figure 1: A refolding between two tetramonohedra

We introduce some key polyhedra. A tetrahedron is a ***tetramonohedron*** if its faces are four congruent acute triangles.[4] We consider a ***doubly covered polygon*** as a special polyhedron with two faces. Precisely, for a given n-gon P, we make a mirror image P' of P and glue corresponding edges. Then we obtain a ***doubly covered n-gon*** which has 2 faces, n edges, and zero volume. A doubly covered rectangle can be regarded as a special case of tetramonohedron whose faces are triangle with a right angle.

3. Refoldabilty of Tetramonohedra and Doubly Covered Triangles

In this section, we first show that any pair of tetramonohedra can be refolded to each other. We note that a doubly covered rectangle is a (degenerate) tetramonohedron, by adding edges along two crossing diagonals (one on the front side and one on the back side). It is known that a polyhedron is a tetramonohedron if and only if it is in Π_4 [4, p. 97]. In other words, Π_4 is the set of tetramonohedra.

Theorem 1. *For any $Q, Q' \in \Pi_4$, Q is refoldable to Q'.*

Proof. Let T be any triangular face of Q. Let a be the length of the longest edge of T and b be the height of T for the base edge of length a. We define T', a', and b' in the same manner for Q'; refer to Figure 1. We assume $a > a'$ without loss of generality. Now we have $a' > b'$ because a' is the longest edge of T', and $a'b' = ab$ because T and T' are of the same area. Thus, $(a')^2 = a'b'\frac{a'}{b'} > a'b' = ab$, and $2a' > a' > b$ by $a > a'$.

We cut two edges of Q of length a, resulting in a cylinder of height b and circumference $2a$. Then we can cut the cylinder by a segment of length $2a'$ because $2a' > b$. The resulting polygon is a parallelogram such that two opposite sides have length $2a$ and the other two opposite sides have length $2a'$(Figure 1). Now we glue the sides of length $2a$ and obtain a cylinder of height b' and circumference $2a'$. Then we can obtain Q' by folding this cylinder suitably (the opposite of cutting two edges of Q' of length $2a'$). □

[4]This notion is also called ***isoscels tetrahedron*** or ***isotetrahedron*** in some literature.

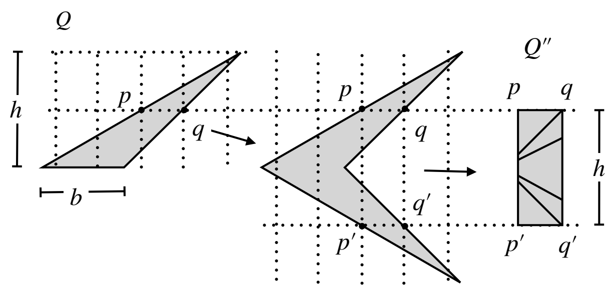

Figure 2: A refolding from a doubly covered triangle to a doubly covered rectangle

To complement the doubly covered rectangles handled by Theorem 1, we give a related result for doubly covered triangles:

Theorem 2. *Any doubly covered triangle Q is refoldable into a doubly covered rectangle. Thus, Q has a refolding sequence to any doubly covered triangle Q' of length at most 3. If doubly covered triangles Q and Q' share at least one edge length, then the sequence has length at most 2.*

Proof. Let Q consist of a triangle T and its mirror image T'. We first cut Q along any two edges, and unfold along the remaining attached edge, resulting in a quadrilateral unfolding as shown in Figure 2. Let b be the length of the uncut edge, which we call the ***base***, and let h be the height of T with respect to the base. Let p and q be the midpoints of the two cut edges. Then the line segment pq is parallel to the base and of length $b/2$. In the unfolding of Q, let p' and q' be the mirrors of p and q, respectively. Then we can draw a grid based on the rectangle $pp'q'q$ as shown in Figure 2. By folding along the crease lines defined by the grid, we can obtain a doubly covered rectangle Q'' of size $b/2 \times h$ (matching the doubled surface area of Q). (Intuitively, this folding wraps T and T' on the surface of the rectangle $pp'q'q$.)

Because a doubly covered rectangle is a special case of tetramonohedra, Q'' is a tetramonohedron. Therefore, the second claim follows from Theorem 1. When Q has an edge of the same length as an edge of Q', as in the third claim, we can cut the other two edges of Q and Q' to obtain the same doubly covered rectangle, resulting in a 2-step refolding sequence. □

The technique in the proof of Theorem 2 works for any doubly covered triangle Q even if its faces are acute or obtuse triangles.

4. Refoldability of a Regular Prismatoid to a Tetramonohedron

In this section, we give a 1-step refolding sequence of any regular prism or prismatoid to a tetramonohedron. We extend the approach of Horiyama and Uehara [8], who showed that the regular icosahedron, the regular octahedron, and the regular hexahedron (cube) can be refolded into a tetramonohedron. As an example, Figure 3 shows their common unfolding for the regular icosahedron.

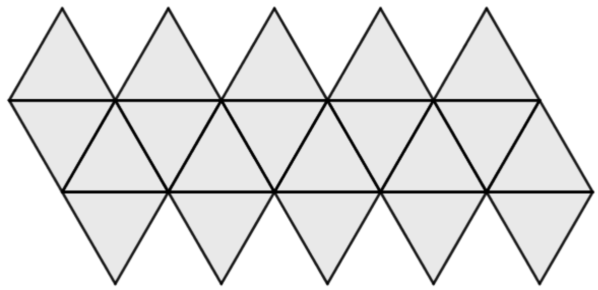

Figure 3: A common unfolding of a regular icosahedron and a tetramonohedron, from [8]

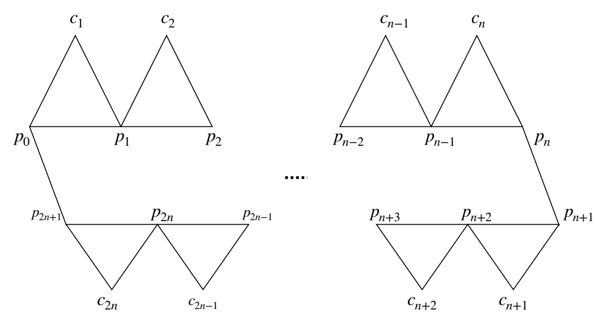

Figure 4: A spine polygon with $2n$ spikes

A polygon $P = (p_0, c_1, p_1, c_2, p_2, \dots, p_{2n}, c_{2n}, p_{2n+1}, p_0)$ is called a ***spine polygon*** if it satisfies the following two conditions (refer to Figure 4):

1. Vertex p_i is on the line segment p_0p_n for each $0 < i < n$; vertex p_i is on the line segment $p_{n+1}p_{2n+1}$ for each $n+1 < i < 2n+1$; and the polygon $B = (p_0, p_n, p_{n+1}, p_{2n+1}, p_0)$ is a parallelogram. We call B the ***base*** of P, and require it to have positive area.
2. The polygon $T_i = (p_i, c_{i+1}, p_{i+1}, p_i)$ is an isosceles triangle for each $0 \le i \le n-1$ and $n+1 \le i \le 2n$. The triangles $T_0, T_1, \dots, T_{n-1}$ are congruent, and $T_{n+1}, T_{n+2}, \dots, T_{2n}$ are also congruent. These triangles are called ***spikes***.

Lemma 2. *Any spine polygon P can be folded to a tetramonohedron.*

Proof. Akiyama and Matsunaga [11] prove that a polygon P can be folded into a tetramonohedron if the boundary of P can be divided into six parts, two of which are parallel and the other four of which are rotationally symmetric. We divide the boundary of a spine polygon P into $l_1 = (p_0, c_1, p_1, c_2, \dots, p_{n-1}, c_n)$; $l_2 = (c_n, p_n)$, $l_3 = (p_n, p_{n+1})$; $l_4 = (p_{n+1}, c_{n+1}, p_{n+2}, c_{n+2}, \dots, p_{2n}, c_{2n})$, $l_5 = (c_{2n}, p_{2n+1})$; and $l_6 = (p_{2n+1}, p_0)$. Then l_3 and l_6 are parallel because the base of P is a parallelogram. Each of l_2 and l_5 is rotationally symmetric on its own as

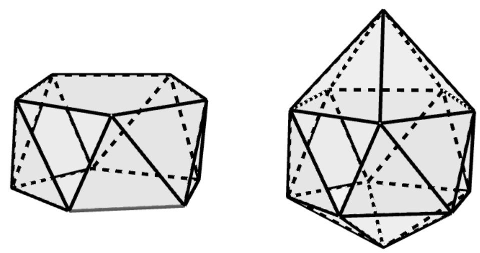

Figure 5: A regular prismatoid and an augmented regular prismatoid

line segments, centering its midpoint. Each of l_1 and l_4 is rotationally symmetric because each spike of P is an isosceles triangle. □

Now we introduce some classes of polyhedra; refer to Figure 5.

A ***prismatoid*** is the convex hull of parallel ***base*** and ***top*** convex polygons. We sometimes call the base and the top ***roofs*** when they are not distinguished. We call a prismatoid ***regular*** if (1) its base P_1 and top P_2 are congruent regular polygons and (2) the line passing through the centers of P_1 and P_2 is perpendicular to P_1 and P_2. (Note that the side faces of a regular prismatoid do not need to be regular polygons.) The perpendicular distance between the planes containing P_1 and P_2 is the ***height*** of the prismatoid. The set of regular prismatoids contains ***prisms*** and ***antiprisms***, as well as doubly covered regular polygons (prisms of height zero).

A ***pyramid*** is the convex hull of a ***base*** convex polygon and an ***apex*** point. We call a pyramid ***regular*** if the base polygon is a regular polygon, and the line passing through the apex and the center of the base is perpendicular to the base. (Note that the side faces of a regular pyramid do not need to be regular polygons.) A polyhedron is an ***augmented regular prismatoid*** if it can be obtained by attaching two regular pyramids to a regular prismatoid base-to-roof, where the bases of the pyramids are congruent to the roofs of the prismatoid and each roof is covered by the base of one of the pyramids.

Theorem 3. *Any regular prismatoid or augmented regular prismatoid of positive volume can be unfolded to a spine polygon.*

Proof. Let Q be a regular prismatoid. Let c_1 and c_2 be the center points of two roofs P_1 and P_2, respectively. Cutting from c_i to all vertices of P_i for each $i = 1, 2$ and cutting along a line joining between any pair of vertices of P_1 and P_2, we obtain a spine polygon. For an augmented regular prismatoid Q, we can similarly cut from the apex c_i of each pyramid to the other vertices of the pyramid, which are the vertices of the roof P_i of the prismatoid. □

When the height of the regular prismatoid is zero (or it is a doubly covered regular polygon), the proof of Theorem 3 does not work because the resulting polygon is not connected. In this case, we need to add some twist.

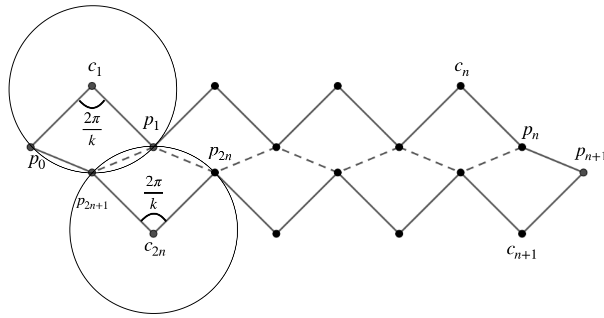

Figure 6: The case of a doubly covered regular 8-gon

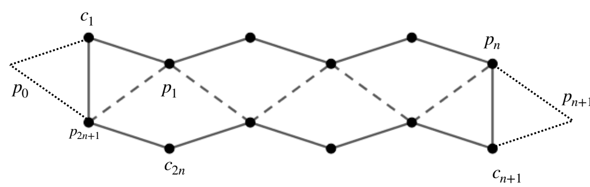

Figure 7: The case of a doubly covered regular 5-gon

Theorem 4. *Any doubly covered regular n-gon is refoldable to a tetramonohedron for $n > 2$.*

Proof. First suppose that n is an even number $2k$ for some positive integer $k > 1$. We consider a special spine polygon where the top angles are $\frac{2\pi}{k}$; the vertices p_0, p_{2n+1}, p_1 are on a circle centered at c_1; and the vertices p_{2n+1}, p_1, p_2 are on a circle centered at c_{2n}; see Figure 6. Then we can obtain a doubly covered n-gon by folding along the zig-zag path $p_{2n+1}, p_1, p_{2n}, p_2, \ldots, p_{n+2}, p_n$ (shown by the doted lines in Figure 6) such that we glue two segments (p_0, p_{2n+1}), (p_n, p_{n+1}) and zip the both side edges at each p_i. Thus when $n = 2k$ for some positive integer k, we obtain the theorem.

Now suppose that n is an odd number $2k+1$ for some positive integer k. We consider the spine polygon whose top angles are $\frac{4\pi}{2k+1}$; the vertices p_0, p_{2n+1}, p_1 are on a circle centered at c_1; and the vertices p_{2n+1}, p_1, p_2 are on a circle centered at c_{2n}. From this spine polygon, we cut off two triangles c_1, p_0, c_{2n+1} and c_{n+1}, p_{n+1}, p_n, as in Figure 7. Then we can obtain a doubly covered n-gon by folding along the zig-zag path $p_{2n+1}, p_1, p_{2n}, p_2, \ldots, p_{n+2}, p_n$ shown in Figure 7. Although the unfolding is no longer a spine polygon, it is easy to see that it can also fold into a tetramonohedron by letting $l'_1 = (c_1, p_1, \ldots, p_n)$, $l'_2 = (p_n, p_n)$, $l'_3 = (p_n, c_{n+1})$, $l'_4 = (c_{n+1}, p_{n+2} \ldots, p_{2n+1})$, $l'_5 = (p_{2n+1}, p_{2n+1})$, and $l'_6 = (p_{2n+1}, c_1)$ in the proof of Lemma 2. $\square$

The proof of Theorem 4 is effectively exploiting that a doubly covered regular $2k$-gon (with $k > 1$) can be viewed as a degenerate regular prismatoid with two k-gon roofs, where each of the side triangles of this prismatoid is on the plane of the roof sharing the base of the triangle.

Because the cube and the regular octahedron are regular prismatoids and the regular icosahedron is an augmented regular prismatoid, we obtain the following:

Corollary 1. *Let Q and Q' be regular polyhedra of the same area, neither of which is a regular dodecahedron. Then there exists a refolding sequence of length at most 3 from Q to Q'. When one of Q or Q' is a regular tetrahedron, the length of the sequence is at most 2.*

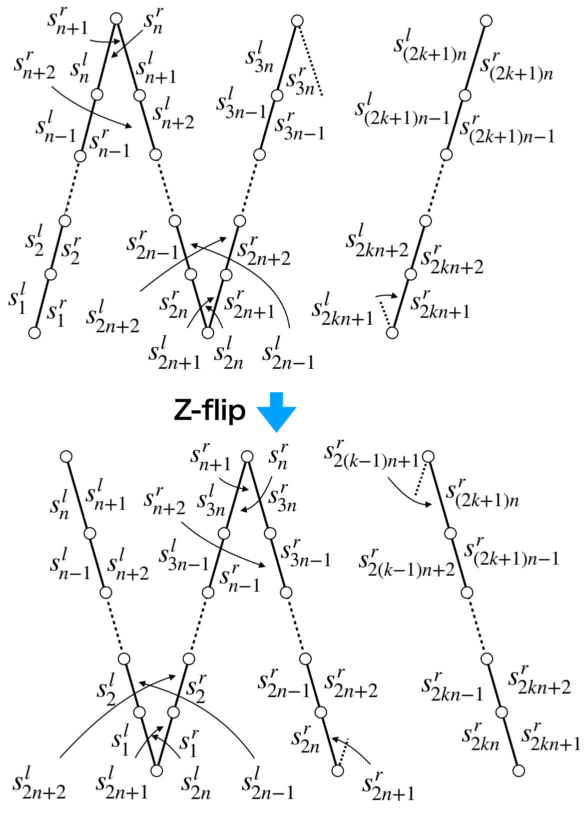

Figure 8: Z-flip

5. Refoldability of a Regular Dodecahedron to a Tetramonohedron

In this section, we show that there is a refolding sequence of the regular dodecahedron to a tetramonohedron of length 4. Combining this result with Corollary 1, we obtain refolding sequences between any two regular polyhedra of length at most 6.

Demaine et al. [6] mention that the regular dodecahedron can be refolded to another convex polyhedron. Indeed, they show that any convex polyhedron can be refolded to at least one other convex polyhedron using an idea called "flipping a Z-shape". We extend this idea.

Definition 1. *For a convex polyhedron Q and $n, k \in \mathbb{N}$, let $p = (s_1, s_2, \ldots, s_{(2k+1)n})$ be a path that consists of isometric and non-intersecting $(2k+1)n$ straight line segments s_i on Q. We cut the surface of Q along p. Then each line segment is divided into two line segments on the boundary of the cut. For each line segment s_i, let s_i^l and s_i^r correspond to the left and right sides on the boundary along the cut (Figure 8). Then p is a **Z-flippable (n,k)-path** on Q, and Q is **Z-flippable** by p, if the following gluing satisfies Alexandrov's conditions.*

- *Glue $s_1^l, s_2^l, \ldots, s_n^l$ to $s_{2n}^l, s_{2n-1}^l, \ldots, s_{n+1}^l$.*
- *Glue $s_1^r, s_2^r, \ldots, s_n^r$ to $s_{2n+1}^l, s_{2n+2}^l, \ldots, s_{3n}^l$.*
- *Glue $s_{n+1}^r, s_{n+2}^r, \ldots, s_{2n}^r$ to $s_{3n}^r, s_{3n-1}^r, \ldots, s_{2n+1}^r$.*

$$\vdots$$

- *Glue $s_{2(k-1)n+1}^r, s_{2(k-1)n+2}^r, \ldots, s_{2kn}^r$ to $s_{(2k+1)n}^r, s_{(2k+1)n-1}^r, \ldots, s_{2kn+1}^r$.*

Figure 9 gives an example of a refolding by a Z-flippable $(1,1)$-path.

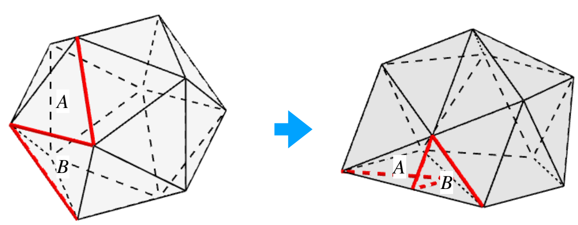

Figure 9: An example of Z-flippable $(1,1)$-paths

*If there are Z-flippable paths $p^1, p^2, \ldots, p^m$ inducing a tree structure on the surface of Q, we can flip them all at the same time (See Figure 10). Then we say that Q is **Z-flippable** by $p^1, p^2, \ldots, p^m$. This method also works when the obtained structure is disconnected trees with no intersections because we can flip each tree independently.*

Theorem 5. *There exists a 4-step refolding sequence between a regular dodecahedron and a tetramonohedron.*

Proof. Let D be a regular dodecahedron. To simplify, we assume that each edge of a regular pentagon is of length 1. We show that there exists a refolding sequence D, Q_1, Q_2, Q_3, Q_4 of length 4 for a tetramonohedron Q_4.

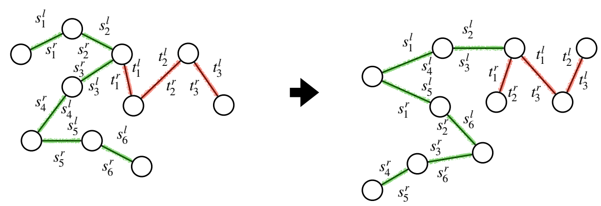

Figure 10: An example of Z-flippable paths that form a tree structure

All cocurvatures of the vertices of D are equal to $\frac{9\pi}{5}$. For any vertices v, there are 3 vertices of distance 1 from v and 6 vertices of distance $\phi = \frac{1+\sqrt{5}}{2}$ from v. Hereafter, in figures, each circle describes a non-flat vertex on a polyhedron and the number in the circle describes its cocurvature divided by $\frac{\pi}{5}$. Each pair of vertices of distance 1 is connected by a solid line, and each pair of vertices of distance ϕ is connected by a dotted line. Figure 11 shows the initial state of D in this notation. We note that solid and dotted lines do not necessarily imply edges (or crease lines) on the polyhedron.

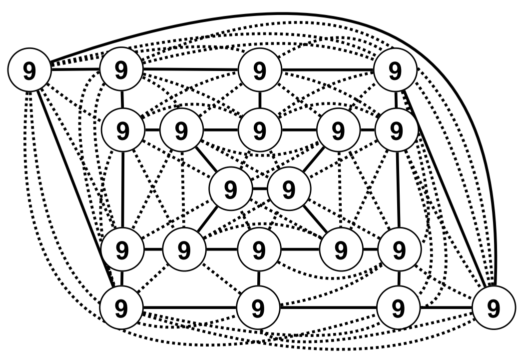

Figure 11: The initial regular dodecahedron

First, we choose $p^1 = (s_1^1, s_2^1, \ldots, s_6^1)$, $p^2 = (s_1^2, s_2^2, s_3^2)$, $p^3 = (s_1^3, s_2^3, \ldots, s_6^3)$, and $p^4 = (s_1^4, s_2^4, s_3^4)$ on the surface of D on the left of Figure 12. Then, p^1 and p^3 are Z-flippable $(2, 1)$-paths and p^2 and p^4 are Z-flippable $(1, 1)$-paths. Thus, D is Z-flippable by p^1, p^2, p^3, p^4 to the polyhedron on the right of Figure 12. Let Q_1 be the resulting polyhedron.

Second, we choose $p^1 = (s_1^1, s_2^1, \ldots, s_5^1)$ on the surface of Q_1 on the left of Figure 13. Then, p^1 is a Z-flippable $(1, 3)$-path. Thus, Q_1 is Z-flippable by p^1 to the next polyhedron Q_2 on the right of Figure 13. Third, we choose $p^1 = (s_1^1, s_2^1, s_3^1)$ and $p^2 = (s_1^2, s_2^2, s_3^2)$ on the surface of Q_2 on the left of Figure 14. Then, p^1 and p^2 are Z-flippable $(1, 1)$-paths. Thus, Q_2 is Z-flippable by p^1 and p^2 to the polyhedron Q_3 on the right of Figure 14.

Fourth, we choose $p^1 = (s_1^1, s_2^1, \ldots, s_5^1)$, $p^2 = (s_1^2, s_2^2, \ldots, s_5^2)$, and $p^3 =$

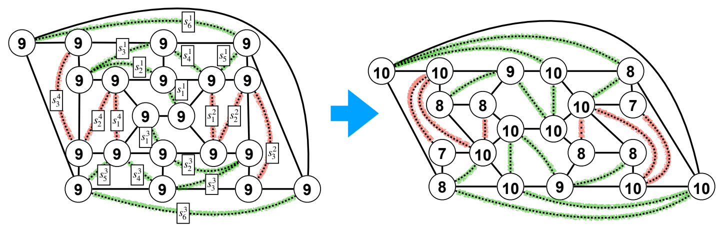

Figure 12: A refolding from D to Q_1

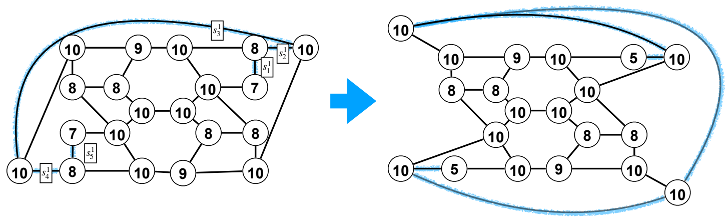

Figure 13: A refolding from Q_1 to Q_2

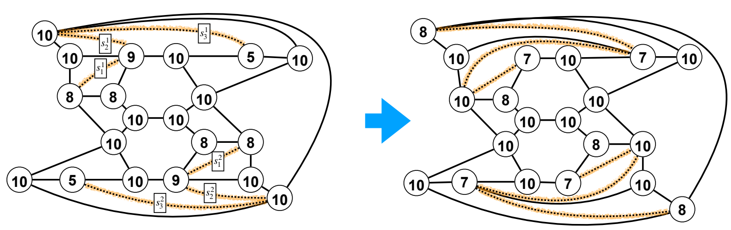

Figure 14: A refolding from Q_2 to Q_3

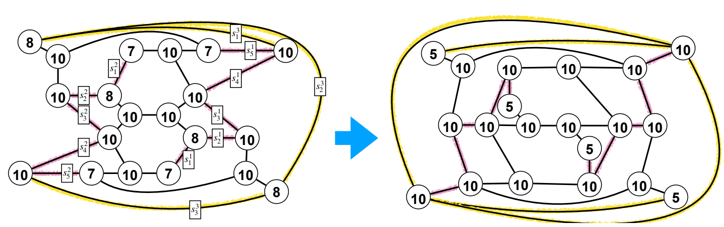

Figure 15: A refolding from Q_3 to Q_4

(s_1^3, s_2^3, s_3^3) on the surface of Q_3 on the left of Figure 15. Then, p^1 and p^2 are Z-flippable $(1,3)$-paths and p^3 is a Z-flippable $(1,1)$-path. Thus, Q_3 is Z-flippable by p^1, p^2, and p^3 to the polyhedron Q_4 on the right of Figure 15. Finally, we obtain a tetramonohedron Q_4 from a regular dodecahedron D by a 4-step refolding sequence.

In this proof, we used partial unfolding between pairs of polyhedra in the refolding sequence. We give the (fully unfolded) common unfoldings in Appendix A. Thus, there exists a 4-step refolding sequence between a regular dodecahedron and a tetramonohedron.

□

6. Refoldability of a Tetrahedron to a Tetramonohedron

In this section, we prove that any tetrahedron can be refolded to a tetramonohedron. Let $\mathcal{Q}_k$ denote the class of polyhedra with exactly k vertices.

Theorem 6. *For any $Q \in \mathcal{Q}_4$, there is at most 3-step refolding sequence from Q to some $Q''' \in \Pi_4$.*

Proof. There are three possible cases about Q: $Q \in \Pi_0$, $Q \in \Pi_1$, or $Q \in \Pi_2$ (the case of $Q \in \Pi_3$ never happen by the Gauss–Bonnet Theorem). First, we consider the case of $Q \in \Pi_0$ and $Q \in \Pi_1$ (See Figure 16). In each case, there are two vertices v_0, v_1 such that $\sigma(v_0) + \sigma(v_1) \leq 2\pi$ by the Gauss–Bonnet Theorem. We cut along the segment v_0v_1 and glue the point v_0 to v_1. On the resulting polyhedron, there are a new vertex v' of a cocurvature $\sigma(v_0) + \sigma(v_1) \leq 2\pi$ and two new smooth vertices λ_0, λ_1. Thus, we can obtain a convex polyhedron $Q' \in \Pi_2$ by Alexandrov's Theorem. That is, we can reduct these two cases to the case of $Q \in \Pi_2$ by the 1-step refolding sequence. Next, we prove that $Q' \in \Pi_2$ is 2-step refoldable into a polyhedron $Q''' \in \Pi_4$ by the following two lemmas.

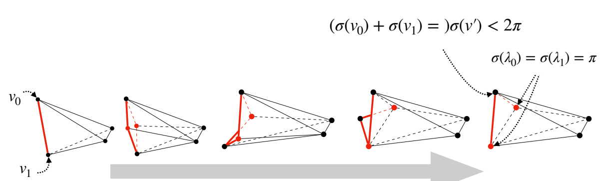

Figure 16: The refolding step Π_0 or Π_1 to Π_2

Lemma 3. *Any polyhedron $Q' \in \Pi_2 \cap \mathcal{Q}_5$ is refoldable into a polyhedron $Q'' \in \Pi_3 \cap \mathcal{Q}_5$.*

Proof. Let λ_0, λ_1 be the two smooth vertices of Q'. We cut Q' by the shortest line segment l joinining λ_0 and λ_1 and denote the obtained surface by C. By making crease lines from each of the other vertices to l perpendicularly and embedding the cut end of C to xy plane, we can form C as a triangular prism sliced (Figure 17). Let $h(t)$ be the hight of a point t on the side of Q'. Let v_0, v_1, v_2 be the other vertices of Q' clockwisely from the viewpoint of the outside of Q' and l_i be the shortest line segments from v_i to v_0. We assume that $h(v_0) \leq h(v_1), h(v_2)$ without loss of generality. Let θ_i denote the angle from the perpendicular line of v_0 to l_i. Then, since $\frac{\pi}{2} \leq \theta_1, \theta_2$ and $\theta_1 + \theta_2 < \sigma(v_0)$, we have $\kappa(v_0) < \pi$.

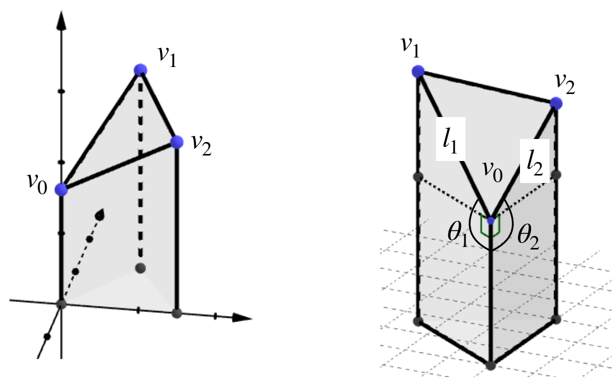

Figure 17: A triangular prism sliced diagonally

Since $\kappa(v_0) + \kappa(v_1) + \kappa(v_2) = 2\pi$ from the Gauss–Bonnet Theorem, at least one of $\kappa(v_1), \kappa(v_2)$ is less than π. Thus we assume $\kappa(v_1) < \pi$. Let l' be the line where the counter-clockwise angle from l_1 to l' at v_0 is π (Figure 18). Note that the clockwise angle from l_1 to l' at v_0 is $\sigma(v_0) - \pi = \pi - \kappa(v_0)$.

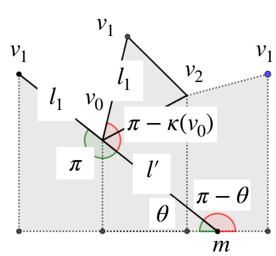

Figure 18: A side view of C

By $h(v_0) \leq h(v_1), h(v_2)$, l' and l have an intersection point m. Let θ be the counter-clockwise angle from l' to l at m. l_1 and l' do not intersect except at v_0 because ${}^\forall t_1 \in l_1, {}^\forall t_2 \in l'$ and $h(t_2) < h(v_0) < h(t_1)$. Then we cut C by l_1, l'(Figure 19).

We denote $\alpha(p)$ as the interior angle of a point p. On the obtained boundary, there are four points whose interior angles are not π: Let p_{v_0}, p_{v_1} correspond to v_0, v_1 and the both of p_m, p'_m correspond to m such that $\alpha(p_{v_1}) =$

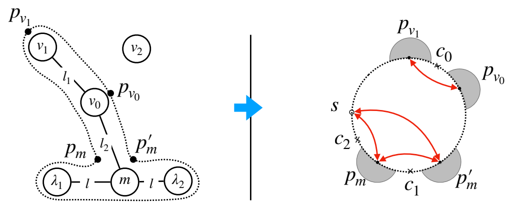

Figure 19: A way of cutting and glueing

$\sigma(v_1), \alpha(p_{v_0}) = \pi - \kappa(v_0), \alpha(p_m) = \theta$, and $\alpha(p'_m) = \pi - \theta$. Let c_0 be the center point of (p_{v_1}, p_{v_0}), and s be the point that have the same distance with p'_m from c_0. Let c_1 and c_2 be the center point of $(p_m, p_{m'})$ and (p_m, s). We glue each of p_{v_0}, p_{v_1} and p'_m, p_m, s. Let Q'' be the resulting polyhedron. Since $\alpha(p_{v_0}) + \alpha(p_{v_1}) = \pi - \kappa(v_0) + \sigma(v_1) = 3\pi - (\kappa(v_0) + \kappa(v_1)) = 3\pi - (2\pi - \kappa(v_1)) = \pi + \kappa(v_1) < 2\pi$. Q'' satisfies the Alexandrov's conditions.
That is, Q'' is a convex polyhedron in $\Pi_3 \cap \mathcal{Q}_5$. □

Lemma 4. *Any polyhedron $Q'' \in \Pi_3 \cap \mathcal{Q}_5$ is refoldable into a polyhedron $Q''' \in \Pi_4$.*

Proof. Let $\lambda_0, \lambda_1, \lambda_2$ be the three smooth vertices of Q'' and v_0, v_1 be the other vertices. In the proof of Lemma 3, the vertex v_3 of Q' remains as the vertex of Q' without cutting, and v_0, v_1 are chosen such that $\kappa(v_0), \kappa(v_1) < \pi$ holds. Thus, we can apply the proof of Lemma 3 to a proof of Lemma 4 by replacing v_2 to λ_2. As a result, we obtain a polyhedron Q''' of Π_4. □
By Lemmas 3 and 4, Theorem 6 follows. □

7. Conclusion

In this paper, we give a partial answer to Open Problem 25.6 in [1]. For every pair of regular polyhedra, we obtain a refolding sequence of length at most 6. Although this is the first refolding result for the regular dodecahedron, the number of refolding steps to other regular polyhedra seems a bit large. Finding a shorter refolding sequence than Theorem 5 is an open problem.

The notion of refolding sequence raises many open problems.

- What pairs of convex polyhedra are connected by a refolding sequence of finite length?
- Is there any pair of convex polyhedra that are not connected by any refolding sequence?

At the center of our results is that the set of tetramonohedra induces a clique by the binary relation of refoldability.

- Is the regular dodecahedron refoldable to a tetramonohedron?

- Are all Archimedean and Johnson solids refoldable to tetramonohedra?
- Is there any convex polyhedron not refoldable to a tetramonohedron? (If not, we would obtain a 3-step refolding sequence between any pair of convex polyhedra.)

Another open problem is the extent to which allowing or forbidding overlap in the common unfoldings affects refoldability. While we have defined refoldability to allow overlap, in particular to follow [6] where it may be necessary, most of the results in this paper would still apply if we forbade overlap. For example, Appendix A confirms this for our refolding sequence from the regular dodecahedron to a tetramonohedron; while the general approach of Theorem 6 is likely harder to generalize.

- Are there two polyhedra that have a common unfolding but all such common unfoldings overlap? (If not, the two notions of refolding are equivalent.)

Acknowledgments

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Appendix A. Common Unfoldings from Regular Dodecahedron to Tetramonohedron in Theorem 5

Figures A.1, A.2, A.3, and A.4 show the common unfoldings of each consecutive pair of polyhedra in the refolding sequence from the proof of Theorem 5.

[1] E. D. Demaine and J. O'Rourke. ***Geometric Folding Algorithms: Linkages, Origami, Polyhedra***. Cambridge University Press, 2007.

[2] D. Xu, T. Horiyama, T. Shirakawa, and R. Uehara. Common Developments of Three Incongruent Boxes of Area 30, ***COMPUTATIONAL GEOMETRY: Theory and Applications***, Vol. 64, pp. 1–17, August 2017.

[3] T. Shirakawa, T. Horiyama, and R. Uehara. On Common Unfolding of a Regular Tetrahedron and a Cube (in Japanese), ***Journal of Science Origami***, Vol. 4, No. 1, pp. 45–54, 2015. (See [5] for the details.)

[4] Altshiller-Court, N. Modern Pure Solid Geometr. ***Chelsea***, 1979.

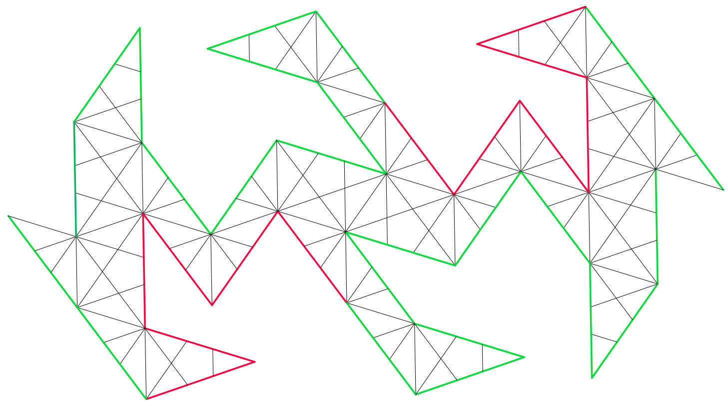

Figure A.1: A common unfolding of D and Q_1

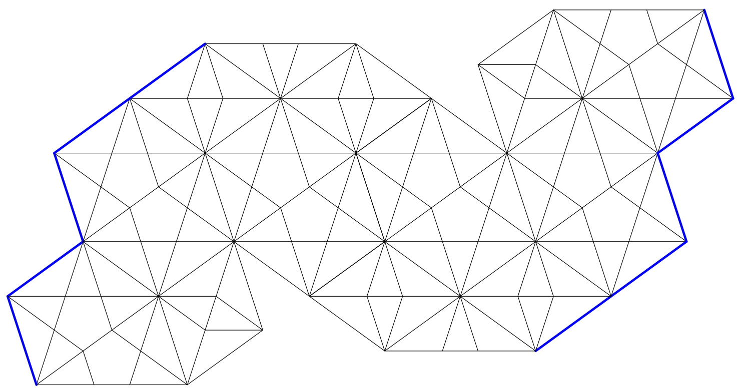

Figure A.2: A common unfolding of Q_1 and Q_2

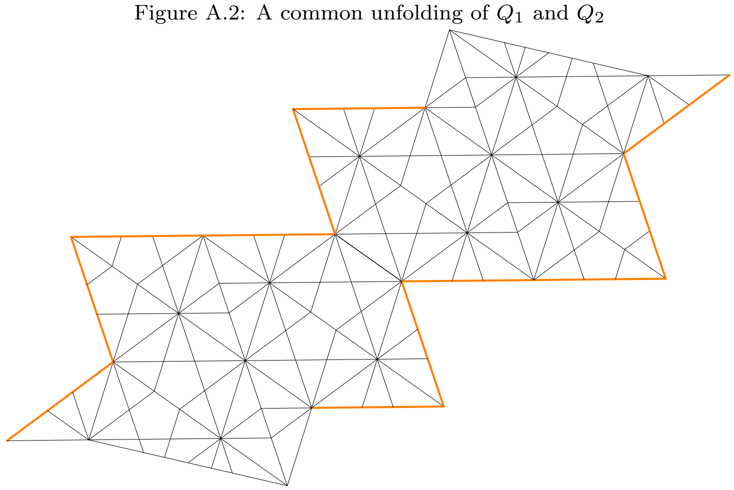

Figure A.3: A common unfolding of Q_2 and Q_3

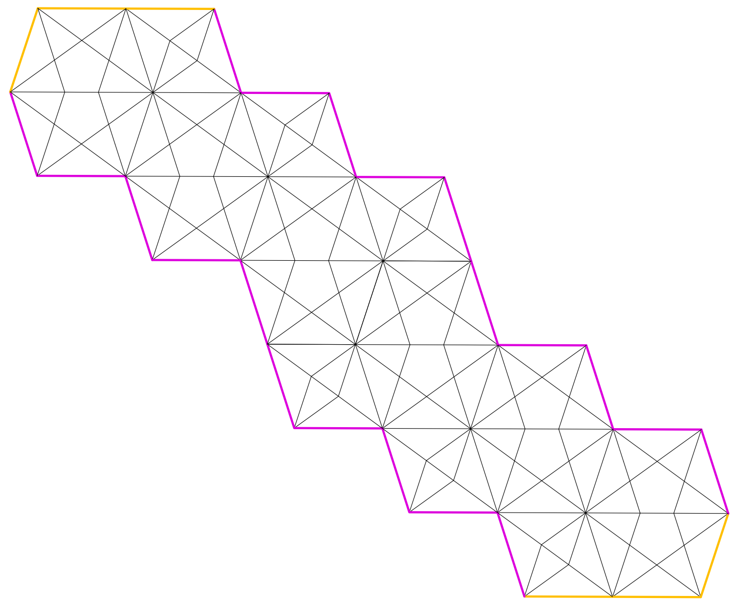

Figure A.4: A common unfolding of Q_3 and Q_4

[5] R. Uehara. ***Introduction to Computational Origami: The World of New Computational Geometry***, Springer, August 2020.

[6] E. D. Demaine, M. L. Demaine, J. Itoh, A. Lubiw, C. Nara, and J. O'Rourke. Refold rigidity of convex polyhedra, ***Computational Geometry***, Vol. 47, pp. 507–517, April 2014.

[7] T. Kamata, A. Kadoguchi, T. Horiyama and R. Uehara. Efficient Folding Algorithms for Regular Polyhedra, ***CCCG 2020***, pp. 131–137, 2020.

[8] T. Horiyama and R. Uehara. Nonexistence of Common Edge Developments of Regular Tetrahedron and Other Platonic Solids. ***China-Japan Joint Conference on Computational Geometry, Graphs and Applications (CGGA 2010)***, pp. 56–57, 2010.

[9] Y. Araki, T. Horiyama, and R. Uehara. Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid, ***Journal of Graph Algorithms and Applications***, Vol. 20, No. 1, pp. 101–114, 2016.

[10] A. S. Biswas and E. D. Demaine. Common Development of Prisms, Anti-Prisms, Tetrahedra, and Wedges, ***CCCG 2017***, pp. 202–207, 2017.

[11] J. Akiyama, K. Matsunaga. An Algorithm for Folding a Conway Tile into an Isotetrahedron or a Rectangle Dihedron, ***Journal of Information Processing***, Vol. 28, pp. 750–758, December 2020.

[12] Alexandr D. Alexandrov. Existence of a convex polyhedron and a convex surface with a given metric. In Yu. G. Reshetnyak and S. S. Kutateladze, editors, ***A. D. Alexandrov: Selected Works***, Part I, pp. 169–173, Gordon and Breach, Australia, 1996.