Tiling with Three Polygons is Undecidable

Erik D. Demaine[*] Stefan Langerman[†]

Abstract

We prove that the following problem is co-RE-complete and thus undecidable: given three simple polygons, is there a tiling of the plane where every tile is an isometry of one of the three polygons (either allowing or forbidding reflections)? This result improves on the best previous construction which requires five polygons.

"Three Rings for the Elven-kings under the sky, …"
— J. R. R. Tolkien, *The Lord of the Rings*, epigraph

1 Introduction

A ***tiling*** of the plane [GS87] is a covering of the plane by nonoverlapping polygons called ***tiles***, isometric copies of one or more geometric shapes called ***prototiles***, without gaps or overlaps. In this paper, we study the most fundamental computational problem about tilings:

Problem 1 (Tiling)**.** Given one or more prototiles, can they tile the plane?

The tiling problem is ***undecidable*** — solved by no finite algorithm. Golomb [Gol70] was first to prove this result, by building n polyominoes that simulate n ***Wang tiles*** [Wan61] — unit squares with edge colors that must match — by adding color-specific bumps and dents to each edge. Four years earlier, Berger [Ber66] proved that tiling with Wang tiles is undecidable (disproving Wang's original conjecture [Wan61]) by showing how they can simulate a Turing machine. Robinson [Rob71] later simplified Berger's proof. The worst-case number n of tiles (Wang or polyomino) is $\Theta(|Q| \cdot |\Sigma|)$, where $|Q|$ and $|\Sigma|$ are the number of states and symbols in the simulated Turing machine, respectively.

Constant Number of Prototiles. The first constant and previously best upper bound on the number of prototiles required to make the tiling problem undecidable is 5, as proved by Ollinger fifteen years ago [Oll09]. Our main result is an improvement of this upper bound to 3:

Theorem 1.1. *Given three simple-polygon prototiles, determining whether they tile the plane is undecidable.*

It remains open whether tiling with one or two given prototiles is decidable. ***Periodic*** tilings (tilings with two translational symmetries) can be found algorithmically by enumerating fundamental domains. (Surprisingly, this intuitive fact does not seem to have been explicitly proved before, except in special settings like Wang tiles [Wan61].) Thus a necessary condition for undecidability is the existence of prototile(s) with only aperiodic tilings. Recently, Smith, Myers, Kaplan, and Goodman-Strauss [SMKGS23] found a single prototile with this property, so there are no obvious obstacles to undecidability.

Tiling by Translation. Our construction relies on rotation of the prototiles (but works independent of whether we allow reflections). If we restrict to tiling by translation only, then Ollinger's construction can be modified to use 11 prototiles, by adding some rotations of the five polyominoes [Oll09]. This upper bound was improved to 10 by Yang [Yan23] and to 8 by Yang and Zhang [YZ24a]. All of these constructions use polyominoes. In higher dimensions, Yang and Zhang [YZ24b] improved the upper bound to five polycube prototiles in 3D, and four polyhypercube prototiles in 4D.

The tiling-by-translation problem also has a lower bound of 2 for undecidability: any single polygon that tiles the plane by translation can do so by periodic (even isohedral) tiling [GBN91]. This result also holds for disconnected polyominoes [Bha20]. If we generalize to tiling a specified periodic subset of d-dimensional space, where d is part of the input, then Greenfeld and Tao [GT24] recently proved tiling to be undecidable with a single disconnected polyhypercube.

Periodic Target. We show that Greenfeld and Tao's generalization to tiling a specified periodic subset [GT24] changes the best known results also for undecidability of tiling the plane. Our 3-polygon construction and Ollinger's 5-polyomino construction [Oll09], and Yang and Zhang's 8-polyomino translation-only polyomino construction [YZ24a] all have one prototile (our shurikens, and their jaws) that appear periodically in any tiling of the plane. Thus, if we remove that pattern from

[*]Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, USA. edemaine@mit.edu

[†]Computer Science Department, Université libre de Bruxelles, Belgium. stefan.langerman@ulb.ac.be

the target, we obtain a periodic subset of the plane which can be tiled using a reduced number of prototiles of 2, 4, and 7, respectively. In particular, we prove

Corollary 1.2. *Given two simple-polygon prototiles, and given a periodic subset of the plane, determining whether the two prototiles tile the periodic subset is undecidable.*

Logical Undecidability. Algorithmic undecidability implies ***logical undecidability*** (as explained in [GT23] in the context of tilings). In particular, our result implies that there are three polygon prototiles that cannot be proved or disproved to tile the plane, for any fixed set of axioms (e.g., ZFC). Otherwise, we would obtain a finite algorithm to decide tileability, by enumerating all proofs.

Corollary 1.3. *For any fixed set of axioms, there are three fixed simple-polygon prototiles such that both "these prototiles tile the plane" and "these prototiles do not tile the plane" have no proof.*

Tiling Completion. Undecidability of tiling requires the set of prototiles to depend on the Turing machine simulation. To obtain undecidability with a fixed set of prototiles, we can generalize the tiling problem [Rob71]:

Problem 2 (Tiling Completion)**.** Given one or more prototiles, and given some already placed tiles, can this placement be extended to a tiling of the plane?

Robinson [Rob71] gave the first result on this problem: a set of 36 prototiles (Wang tiles or polygons) for which tiling completion is undecidable. This result applies the general Turing machine simulation to Minsky's 4-symbol 7-state universal Turing machine, so only a finite number of tiles need to be preplaced to represent the Turing machine to simulate. Likely this result could be improved using newer smaller universal Turing machines [WN09]. If we allow for (countably) infinitely many tiles to be preplaced, we can use semi-universal Turing machines and simulate Rule 110, enabling undecidability with just six supertiles (Wang tile or polygons) [Yan13].

Our main result reduces this upper bound to 3, in the stronger model of finitely many preplaced tiles:

Corollary 1.4. *There are three fixed simple-polygon prototiles such that, given a finite set of already placed tiles, determining whether this placement can be extended to a tiling of the plane is undecidable.*

Co-RE-completeness. While past results on tiling and tiling completion have focused on undecidability, all such proofs actually show ***co-RE-hardness***: the simulated Turing machine halts if and only if the prototiles fail to tile. Co-RE-hardness is a more precise statement than undecidability, so we use that phrasing here. But it also raises the question: are tiling and tiling completion in co-RE? Surprisingly, this question does not seem to have been solved (or even asked) in the literature before. We prove that the answer is "yes":

Theorem 1.5. *Given a finite set of polygon prototiles, and given a (possibly empty) connected set of already placed tiles, determining whether this placement can be extended to a tiling the plane is in co-RE.*

This result holds in a very general model for polygons: the angles and edge lengths can be represented as ***computable*** numbers (meaning that a Turing machine can output the first n bits, given n). Our three-polygon construction uses a more restricted model, where the angles are rational multiples of π and the edge lengths are constant-size radical expressions, showing the problem to be co-RE-complete for every model in between.

Corollary 1.6 (Stronger form of Theorem 1.1)**.** *Given three simple-polygon prototiles, where the angles and edge lengths are specified by computable numbers or by constant-size radical expressions, determining whether they tile the plane is co-RE-complete.*

In this abstract, we sketch the proof of Theorem 1.1. See the full paper [DL24] for full proofs.

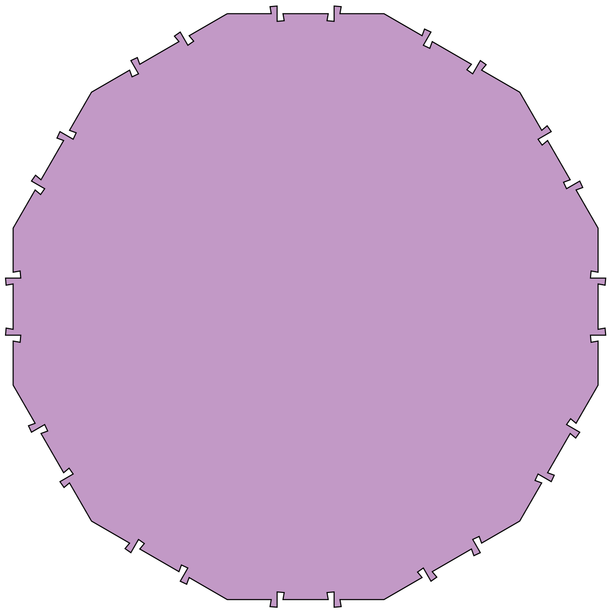
2 Signed Wang Tiling

We reduce from Wang tiling, which is known to be undecidable. Specifically, we use a variation called ***signed free Wang tiling***, where each ***Wang tile*** is a square with a glue on each edge, each ***glue*** has a sign (+ or −) and a value, two glues match exactly if they have opposite sign and equal value, and Wang tiles can be freely translated and/or rotated (but not reflected). In 1971, Robinson [Rob71, p. 179] proved undecidability of tiling the plane with a square grid of such Wang tiles and matching glues, via a simple reduction from the (unsigned translation-only) Wang tiling problem proved undecidable by Berger [Ber66]. These results in fact establish co-RE-hardness.

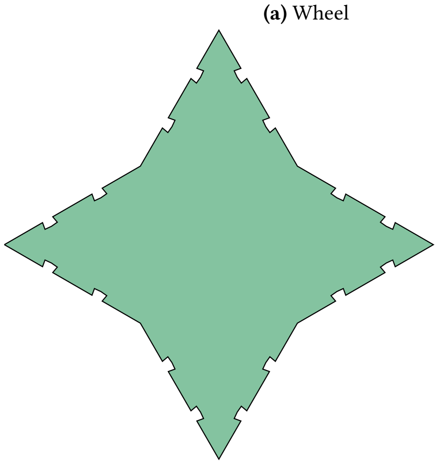
3 Three Tiles That Simulate *n* Signed Wang Tiles

We implement any set of n (signed free) Wang tiles with three tiles, illustrated in Figure 1:

1. the *wheel* which encodes all of the Wang tiles,
2. the *staple* which covers the unused Wang tiles of each wheel, and
3. the *shuriken* which fills in the remaining gaps.



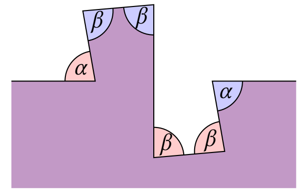
(a) Wheel



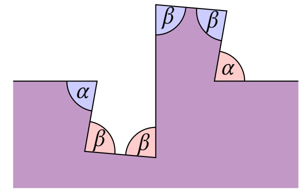
(b) Shuriken

(c) Staple

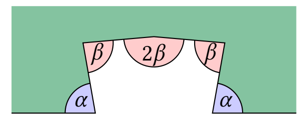
Figure 1: The three tiles in our construction, to scale; Figure 2 shows zoomed details of the construction. The wheel is just an example; it depends on the n Wang tiles being simulated. The shuriken depends (only) on n.



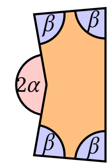
(a) Wheel: Tweedledee (0)



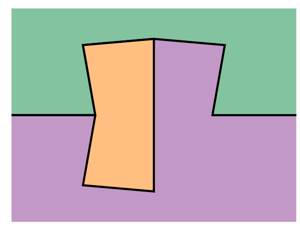
(b) Wheel: Tweedledum (1)



(c) Shuriken: Notch



(d) Staple



(e) Combining the tweedle, notch, and staple.

Figure 2: Zoomed views of portions of the three tiles in our construction (10× scale compared to Figure 1).

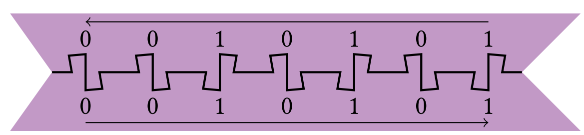


Figure 3: Matching a glue (top) and its negative (bottom) between two wheels.

Suppose we are given a set of n Wang tiles, where the ith tile ($1 \le i \le n$) has signed glues n_i, e_i, s_i, w_i on its north, east, south, and west edges respectively. Assume n is an odd integer ≥ 5 by possibly adding duplicate tiles.

The ***wheel*** is a regular $4n$-gon with each edge adorned by bumps and notches representing the $4n$ glues. For tile i, the glues n_i, e_i, s_i, w_i adorn sides $i, n+i, 2n+i, 3n+i$ of the $4n$-gon, respectively. To encode a glue, we encode its value in binary using $b = O(\log n)$ bits, prepend a 00 at the beginning, and append 01 at the end. For negative glues, we reverse the order of the bits, which puts a 10 at the beginning and a 00 at the end. Then we represent each bit with a ***tweedledee*** (0) or ***tweedledum*** (1) gadget, which are rotationally symmetric zig-zags shown in Figures 2(a) and 2(b). Both follow the sequence of angles $\alpha, \beta, \beta, \beta, \beta, \alpha$ where $\alpha = \frac{\pi}{2} - 2\varepsilon$, $\beta = \frac{\pi}{2} - \varepsilon$, and $\varepsilon = \frac{\pi}{16}$. For tweedledee, this sequence measures defect, angle, angle, defect, defect, angle, respectively; while for tweedledum, this measures the opposite (angle, defect, defect, angle, angle, defect). As shown in Figure 3, two adjacent glues match exactly if and only if they have the same value and opposite sign (where the opposite sign is enforced by the 00 and 01 at either end). This representation also ensures that reflecting a wheel will produce reflected glues that do not match unreflected glues: a reflection causes the bits of a glue to be reversed and negated, so the reflection of a positive glue starts with 01 and ends with 11, and the reflection of a negative glue starts with 11 and ends with 10, both of which are incompatible with unreflected glues.

By this construction, rotating the wheel so that its ith

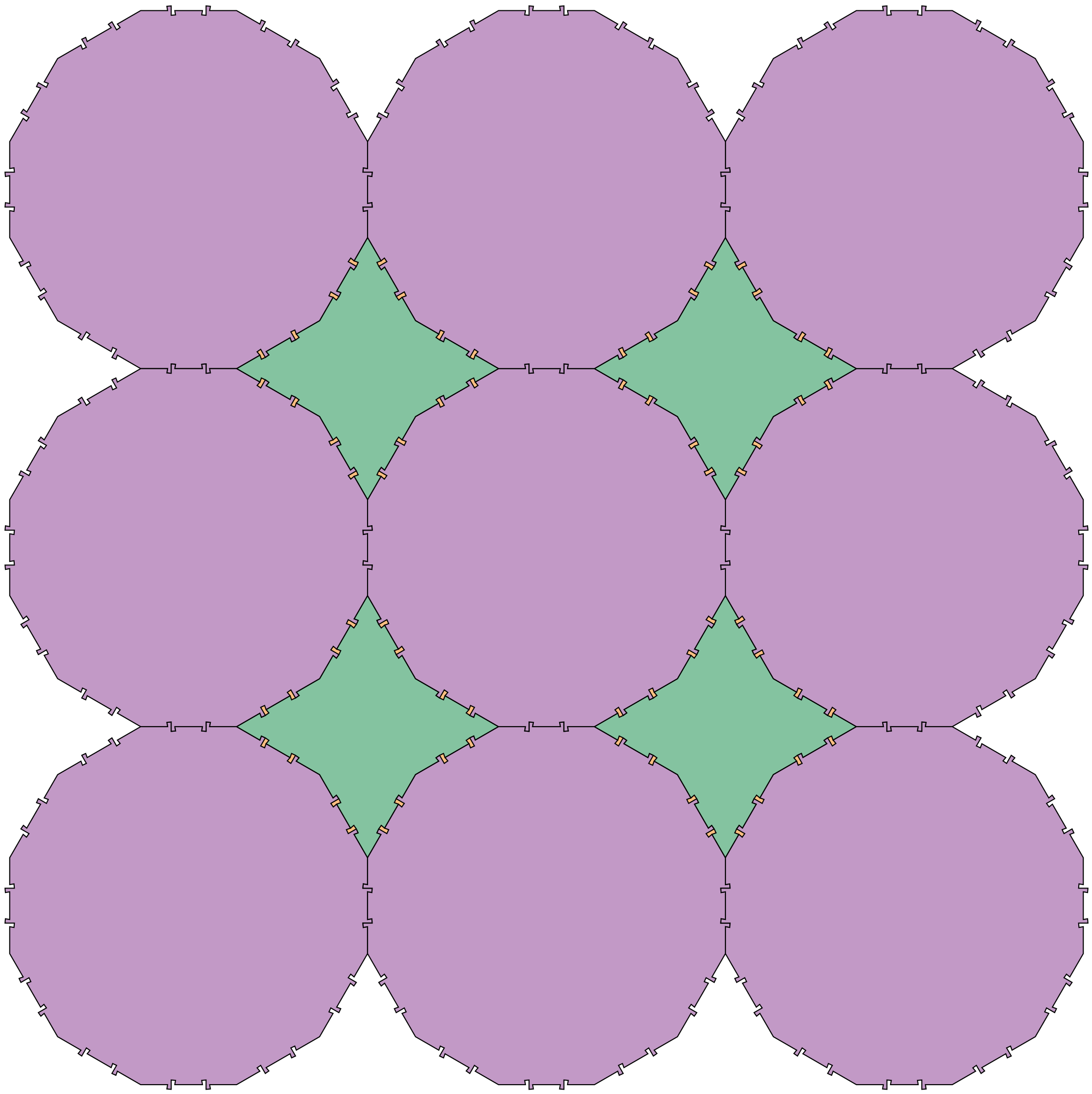


Figure 4: Example tiling with the wheel, shuriken, and staple.

side is horizontal and at the top will have its north, east, south, and west sides represent the glues n_i, e_i, s_i, w_i of tile i. Given a tiling of the plane using this set of Wang tiles, we can place copies of the rotated wheel exactly as in the Wang tiling, and the glues will match exactly. Some space remains between the wheels, which we fill with "staples" and "shurikens". See Figure 4.

The ***shuriken*** is composed of four regular concave chains of $n-1$ sides, matching the lengths and complementary to the angles of the regular $4n$-gon. Each side is adorned with b reflectionally symmetric ***notches***, shown in Figure 2(c), each consisting of convex angle α; reflex deficits $\beta, 2\beta, \beta$; and convex angle α. As shown in Figure 2(e), each notch can fit a tweedle of either kind, leaving a space that is filled exactly by a ***staple*** (shown in Figure 2(d), and consisting of convex angles $\beta, \beta, \beta, \beta$ and reflex deficit 2α). Thus each side of the shuriken can exactly match any glue, effectively hiding the unused tiles of each wheel (the glues that are not on the north, east, south, or west sides).

Thus we have shown one direction of the reduction: given a set of n Wang tiles and a tiling of the plane with them, we can construct a tiling of the plane with the wheel, the shuriken, and the staple. To show that this intended tiling is the only way our three tiles can tile the plane, we analyze the limited ways in which the angles of the shapes can fit together. In particular, we prove that staples alone, then staples and shurikens together, cannot tile the plane. This guarantees the existence of a wheel, and then we show that it must be surrounded by an alternation of shurikens and staples, which eventually forces a Wang tiling.

Acknowledgements

This research was initiated at the Second Adriatic Workshop on Graphs and Probability, and continued at the Eleventh Annual Workshop on Geometry and Graphs. The authors thank all participants of both workshops for providing a stimulating research environment.

References

[Ber66] Robert Berger. The undecidability of the domino problem. *Memoirs of the American Mathematical Society*, 66:1–72, 1966.

[Bha20] Siddhartha Bhattacharya. Periodicity and decidability of tilings of $\mathbb{Z}^2$. *American Journal of Mathematics*, 142(1):255–266, 2020.

[DL24] Erik D. Demaine and Stefan Langerman. Tiling with three polygons is undecidable. arXiv:2409.11582, 2024. https://arxiv.org/abs/2409.11582.

[GBN91] D. Girault-Beauquier and M. Nivat. Tiling the plane with one tile. In *Topology and Category Theory in Computer Science*. Oxford University Press, August 1991.

[Gol70] Solomon W. Golomb. Tiling with sets of polyominoes. *Journal of Combinatorial Theory*, 9(1):60–71, 1970.

[GS87] Branko Grünbaum and G. C. Shephard. *Tilings and Patterns*. W. H. Freeman, 1987.

[GT23] Rachel Greenfeld and Terence Tao. Undecidable translational tilings with only two tiles, or one nonabelian tile. *Discrete & Computational Geometry*, 70(4):1652–1706, 2023.

[GT24] Rachel Greenfeld and Terence Tao. Undecidability of translational monotilings. *Journal of the European Mathematical Society*, 2024. To appear.

[Oll09] Nicolas Ollinger. Tiling the plane with a fixed number of polyominoes. In *Proceedings of the 3rd International Conference on Language and Automata Theory and Applications (LATA 2009)*, volume 5457 of *Lecture Notes in Computer Science*, pages 638–647, Tarragona, Spain, April 2009. Springer.

[Rob71] Raphael M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12(3):177–209, September 1971.

[SMKGS23] David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss. An aperiodic monotile. arXiv:2303.10798, 2023. https://arxiv.org/abs/2303.10798.

[Wan61] Hao Wang. Proving theorems by pattern recognition — II. *Bell Systems Technical Journal*, 40(1):1–41, 1961.

[WN09] Damien Woods and Turlough Neary. The complexity of small universal turing machines: A survey. *Theoretical Computer Science*, 410(4):443–450, 2009.

[Yan13] Jed Chang-Chun Yang. *Computational Complexity and Decidability of Tileability*. PhD thesis, University of California, Los Angeles, 2013.

[Yan23] Chao Yang. Tiling the plane with a set of ten polyominoes. *International Journal of Computational Geometry & Applications*, 33(03n04):55–64, 2023.

[YZ24a] Chao Yang and Zhujun Zhang. A proof of Ollinger's conjecture: undecidability of tiling the plane with a set of 8 polyominoes. arXiv:2403.13472, 2024. https://arxiv.org/abs/2403.13472.

[YZ24b] Chao Yang and Zhujun Zhang. Undecidability of translational tiling of the 4-dimensional space with a set of 4 polyhypercubes. arXiv:2409.00846, 2024. https://arxiv.org/abs/2409.00846.