Folding Triangular and Hexagonal Mazes

Erik D. Demaine, Jason S. Ku, Madonna Yoder

Abstract:
We present algorithms to fold a convex sheet of paper into a maze formed by extruding from a floor, a subset of edges from either a regular triangular or hexagonal grid. The algorithm provides constructions which are efficient, seamless, and watertight.

1 Introduction

At 5OSME, [Demaine et al. 10] presented an efficient construction to fold orthogonal mazes, computable in polynomial time. The approach was to produce a constant number of fixed gadgets comprising all possible maze intersections in an orthogonal grid, such that compatible orthogonal boundaries of the gadgets match, allowing their combination into arbitrary configurations. In this paper, we present two similar construction approaches to efficiently fold mazes on both triangular and hexagonal grids.

As in the referenced orthogonal maze construction, our constructions are **efficient**, i.e. foldable from paper that is a small scale factor larger than the final maze, **watertight** [Demaine and Tachi 17], i.e. the boundary of the paper maps to the boundary of the model, and **seamless**, i.e. designated visible faces of the target input are covered by only one uncreased and visible layer of paper in the folding. For these mazes, each cell of the base grid viewed from above, and each rectangular wall between grid points viewed from either direction comprises the seamless faces of our target. While a hexagonal maze could be folded using a triangular maze construction, doing so would not maintain the seamless property on cells of the hexagonal grid, so we present both constructions here.

Whereas orthogonal mazes include vertices with maximum degree four, triangular and hexagonal grids have maximum degrees of six and three respectively. So while the orthogonal construction required six distinct intersection gadgets, a triangular grid requires thirteen gadgets up to reflection and rotation, while a hexagonal grid requires only four.

Similarly to the original orthogonal construction, all of our triangular grid intersection gadgets require only a single pleat of paper to construct a maze wall of height h, achieving conjectured optimal scale factor $2h+1$. However, we show that if maze walls connect at an angle of more than $90°$, additional paper must be used to fold them if the foldings are to remain both seamless and watertight. Thus, the walls of our hexagonal grid gadgets use additional paper in their construction.

2 Algorithm

The algorithm follows the same construction template as [Demaine et al. 10]. A target input maze on a regular grid is specified by labeling grid edges as either extruded or not extruded. To construct the maze, each vertex of the grid is replaced by an appropriate folded gadget corresponding to the extrusion profile of the edges incident to the vertex. By constructing the folded gadgets to expose the boundary of the paper along the boundary of the folding in a consistent way (i.e. symmetric and watertight), these gadgets can be stitched together to form a single sheet folding of the input maze. There are a constant number of vertex gadgets, so selecting the corresponding gadget for each vertex takes constant time, allowing gadgets to be tiled in linear time with respect to the number of grid intersections contained in the grid.

Given a regular grid having vertices of degree d, there are 2^d possible assignments of edges incident to the vertex; however many of the assignments are equivalent up to rotation or reflection. For a rectangular grid, $d = 4$ but there are only six possible extrusion profiles for a vertex, up to reflection and rotation. Thus, [Demaine et al. 10] provided corresponding folded states for six grid gadgets. In this paper, we provide folded gadgets corresponding to vertex extrusion profiles for regular hexagonal and triangular grids. A regular hexagonal grid has degree $d = 3$, admitting only four possible extrusion profiles, while a regular triangular grid has degree $d = 6$, admitting thirteen possible extrusion profiles. These profiles are shown in Figure 1.

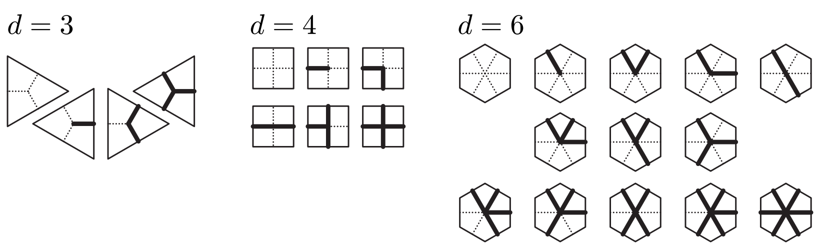

Figure 1: *Extrusion profiles for the three regular grid tilings. Thick black denotes extruded edges, while dotted lines denote not extruded.*

Crease patterns for our four regular hexagonal grid intersection gadgets are shown in Figure 2, while Figure 3 depicts crease patterns for our thirteen regular triangular grid intersection gadgets. Red and blue lines denote mountain folds (folding backward) and valley folds (folding forward) respectively, while line thickness is proportional to fold angle in the folded state. Gray regions denote paper that is not visible from the top surface of the model, while the white regions are the

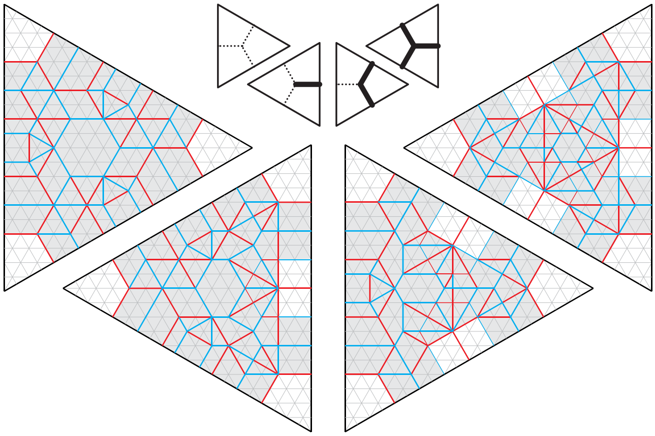

Figure 2: *The four hexagonal grid maze intersection gadgets.*

seamless faces that make up the walls and floor of the folded maze. Similarly to the orthogonal construction of [Demaine et al. 10], these gadgets can be adjusted to allow for arbitrary extrusion height h relative to a normalized unit grid by sinking the gray regions appropriately.

Assuming unit distance between grid points and extrusion height h, each triangular grid gadget can be folded from a hexagon of side length $2h+1$, the same scale factor that was achieved by the orthogonal grid construction (and is within a constant factor of optimal for similar reasons). We suspect our folding is optimal for the complete triangular grid, where all edges are extruded, at least among watertight foldings. The hexagonal grid gadgets achieve strictly worse scale factor $6h+1$, a ratio chosen partially to ensure that crease pattern vertices lie on a grid. Is it possible to achieve scale factor $2h+1$ for hexagonal grid gadgets? In fact, if we constrain our design to produce modular grid-aligned gadgets that are seamless and watertight, the answer is no, a scale factor of $2h+1$ is not achievable.

Theorem 1. *No set of watertight, seamless hexagonal grid-aligned maze gadgets can be achieve a scale factor of* $2h+1$ *or less.*

Proof. Consider the particular hexagonal extrusion profile containing a single extruded edge, and the grid-aligned unfolding of its seamless faces embedded in a paper with scale factor $2h+1$ as shown in Figure 4. The watertight property requires that the boundary of the seamless folded faces, map to the boundary of the unfolded paper, so this unfolding is fixed. Now, consider the distance between the central vertex of the extruded wall (marked with a black circle in the figure) and an edge of the boundary not containing the extruded edge (marked with thick black line). The shortest path between them in the unfolding is shown by the red line,

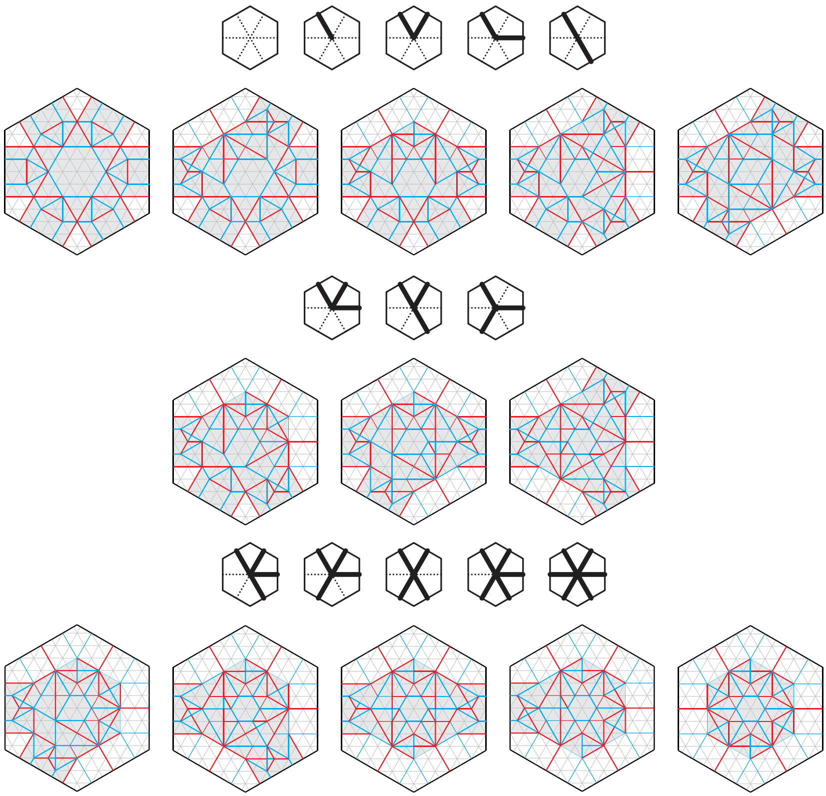

Figure 3: *The thirteen triangular grid maze intersection gadgets.*

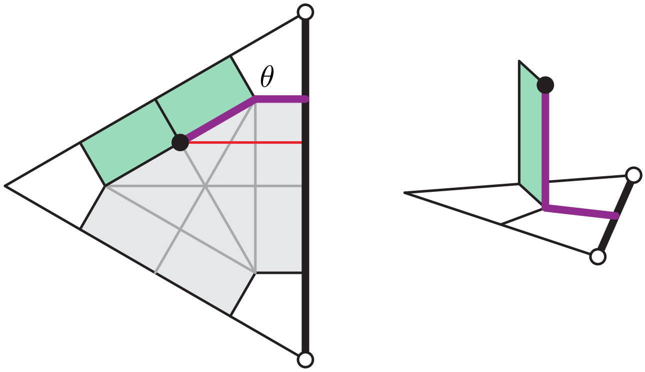

Figure 4: *When θ is larger than $90°$, the distance between the dark point and the marked edge along the boundary in the unfolded paper [Left] is shorter than the intrinsic distance between the marked point and the marked edge in a corresponding folding of the white and green regions [Right]; so if the maze folding is to be watertight, a larger equilateral triangle of paper must be used.*

while a path achieving the intrinsic distance between the black dot and any point on that boundary edge in the folding is marked in purple. The purple path is longer than the red path, thus since folding cannot increase intrinsic distances, no folding exists from a triangle with scale factor $2h+1$ to the desired folding. □

Thus while our hexagonal construction has a lower scale factor than the orthogonal or triangular grid constructions, the same scale factor is not attainable under the specified design constraints. To demonstrate the regular triangular grid gadgets presented in this paper, Figure 5 depicts a folded version of the MIT logo embedded on a triangular grid.

Acknowledgments

This work was initiated during an open problem session associated with MIT class 6.849 (Geometric Folding Algorithms) taught by E. Demaine. We thank the other participants of that class for providing a stimulating research environment.

References

[Demaine and Tachi 17] Erik D. Demaine and Tomohiro Tachi. "Origamizer: A Practical Algorithm for Folding Any Polyhedron." In *Proceedings of the 33rd International Symposium on Computational Geometry (SoCG 2017)*, p. to appear. Brisbane, Australia, 2017.

[Demaine et al. 10] Erik D. Demaine, Martin L. Demaine, and Jason Ku. "Folding Any Orthogonal Maze." In *Origami^5: Proceedings of the 5th International Conference on*

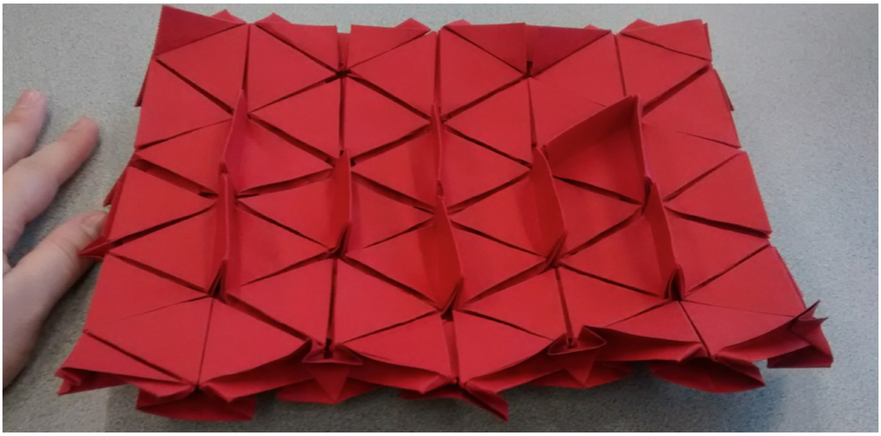

Figure 5: *A triangular grid maze representing the letters MIT folded by M. Yoder using the gadgets and construction described in this paper.*

Origami in Science, Mathematics and Education (OSME 2010), pp. 449–454. Singapore: A K Peters, 2010.

Erik D. Demaine
CSAIL, 32 Vasser St., Cambridge, MA 02139, e-mail: edemaine@mit.edu

Jason S. Ku
MIT, 32 Vasser St., Cambridge, MA 02139, e-mail: jasonku@mit.edu

Madonna Yoder
MIT, 32 Vasser St., Cambridge, MA 02139, e-mail: yodermadonna@gmail.com