Scaling any Surface Down to any Fraction

Erik D. Demaine, Martin L. Demaine, and Kayhan F. Qaiser

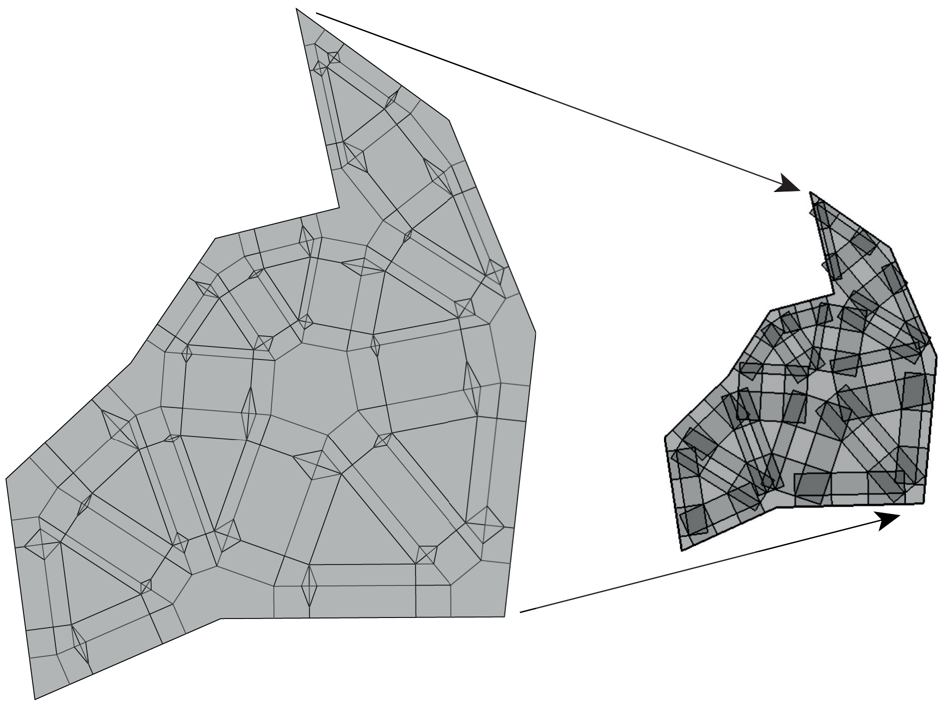

FIGURE 1. Dog head creased and folded to half its size. (Rendered using Tomohiro Tachi's Freeform Origami software, http://www.tsg.ne.jp/TT/software/.)

1. Introduction

In recent years, origami has found a wealth of applications in engineering, science, and manufacturing. One of these application areas is the ability to transform an object into a smaller form, for storage or transportation and possibly later deployment in full scale (deployable structures). Examples include Miura-ori applied to solar cells and collapsible maps/atlases [**Miu06**], Lang and LLNL's eyeglass telescope lens [**Hel03**], airbag folding [**CE06**], and the origami stent graft [**KTY$^+$06**].

In this paper, we explore a precise form of this general problem, where the goal is to fold a given polyhedral shape into a scaled down copy of itself, instead of just any smaller shape. Specifically, we show

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how to fold a given polyhedral surface into itself scaled down by a factor of $\lambda \in [\frac{1}{3}, 1]$ using a new kind of tessellation fold. By repeated application of the same folding, one could in theory fold an object down to any smaller scaling of itself.

This problem is part of a general family of problems that aim to find the most efficient folding for a particular family of constructions. Our approach follows in a similar vein to Maze Folding [**DDK10**]. By breaking the problem into simplified subunits, we obtain a very general result. In our case, the subunits are acute triangles further divided into three quadrilaterals. We specify the folding for each quadrilateral unit separately, and then merge the units together, showing there are no inconsistencies between units during merging.

The folding of each quadrilateral is a simple flat twist fold as defined by Lang and Batemen in [**LB10**] as "a construction composed of a polygon (usually, but not always, regular) with pleats radiating away from the polygon". The creasing results in a folded form of smaller surface area than the original and with the inner polygon rotated by some angle between 0 and 180°. The folding is usually accompanied by a hinging motion on the vertices of the inner polygon. Because our polygons are rotated by 180°, they can be thought of as the special case of the simple twist, as they do not twist but rather perform two flips or reflections. We will thus refer to the maneuver as a double flip.

The construction provides a new way to create tessellations. It is worth noting that tessellations themselves are fairly old. Their original ancestor were likely pleated fabrics, which appeared in Europe in the 20th century (or earlier). Twist-folds were popularized in Japan by Shuzo Fujimoto, who self-published the first origami tessellation book Twist Origami in 1976. The twist-fold provided an alternative to folding representational objects like animals as the new folding technique opened up fascinating geometric possibilities which naturally led to tessellations. Ron Resch had previously patented designs of three dimensional tessellations in 1966 [**ao68**]. He was followed by David Huffman who pioneered work on curved crease sculptures [**Huf76**]. In more modern times Chris K. Palmer [**Pal**] and Eric Gjerde [**Gje09**] have explored the artistic potential of the twist and tessellations. The field has academic, artistic and practical avenues awaiting exploration.

2. Algorithm

2.1. Overview. The high level steps of the algorithm are outlined below:

(1) Divide the surface into acute triangles.
(2) Overlay each triangle's Voronoi diagram to further divide the polygon into quadrilaterals.
(3) Crease a double flip on each quad.
(4) Merge quads into triangles and then triangles into original surface.
(5) Fold.

2.2. Acute Triangulation. The starting point for our algorithm is an acutely triangulated surface. While it is easy to triangulate a polygon or polyhedral surface, creating an acute triangulation is a challenging problem popularized by Martin Gardner [**Gar95**]. If the surface is a single polygon with n sides, Maehara [**Mae02**] proved that $O(n)$ acute triangles suffice, and the constant was later improved by Yuan [**Yua05**]. For general polyhedral surfaces, existence of acute triangulations was proved by Burago and Zalgaller [**BZ60**] and later simplified by Saraf [**Sar09**]. Unfortunately, neither result gives a good bound on the number of required triangles. Furthermore, both results give only a *geodesic* triangulation, so the edges of the triangulation are shortest paths which may cross the edges of the polyhedron. Fortunately, such a geodesic triangulation suffices for our purposes, as we can conceptually fold each triangle separately to a scaled down copy of itself, then bend it to match the 3D geometry of the given polyhedron.

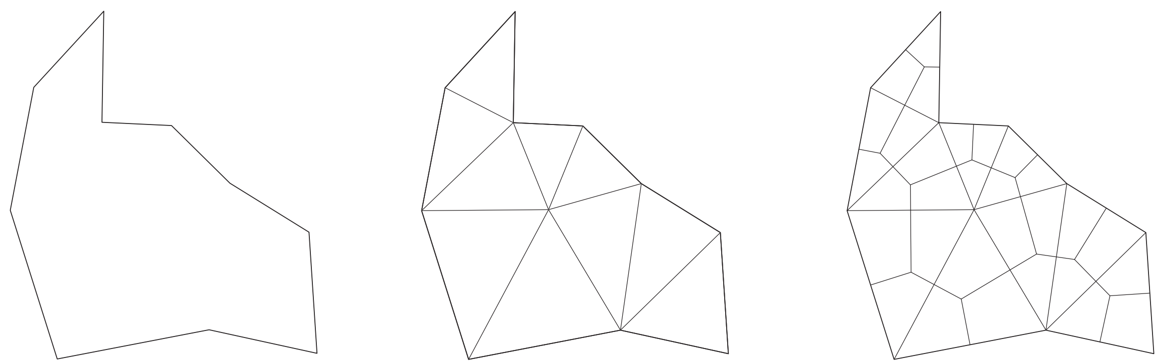

FIGURE 2. Triangulation and Voronoi diagram

2.3. Voronoi Diagram. We divide each triangle into 3 quadrilaterals by extending lines from the circumcenter to each side. This is equivalent to dividing the triangulation along its Voronoi diagram. Because every triangle is acute the circumcenter will lie inside it. It is here that we need an acute triangulation of the surface.

The quadrilaterals created in the this step are the units we will fold. It is worth noting some properties of these quadrilaterals as shown in figure 3. Each quad has an obtuse angle at O and an acute angle at V. The other two vertices are right angles. Each quad lies within the circle with diameter from O to V because the other two angles are right and applying Thales' Theorem. In addition each quad also must have a circumcenter lying at the midpoint of OV because each quad is comprised of two right angled triangles joined on their hypotenuses.

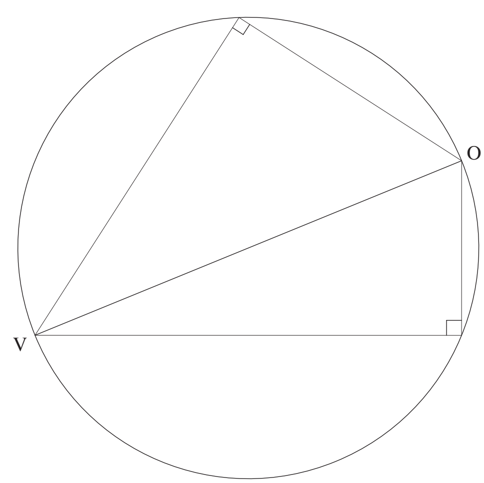

FIGURE 3. Space in which quads exist.

2.4. Double Flip. We will now begin to outline the double flip construction and the constraints required to make it a valid folding. Before introducing the general double flip, let us consider the special case where the quadrilateral is a square. Although our algorithm would never produce this case, the generalized folding follows from it. As shown in figure 4, the folding for the square is simply two pleats that are perpendicular to each other. The darker creases are mountains and the lighter creases are valleys. The foldings in figure 4 are for scale factor $\lambda = 1/2$, where the lengths of the surface shrink by λ and the pleat lengths are $\lambda/2$. It is also important to note that the square and the quad are simply transformed versions of each other.

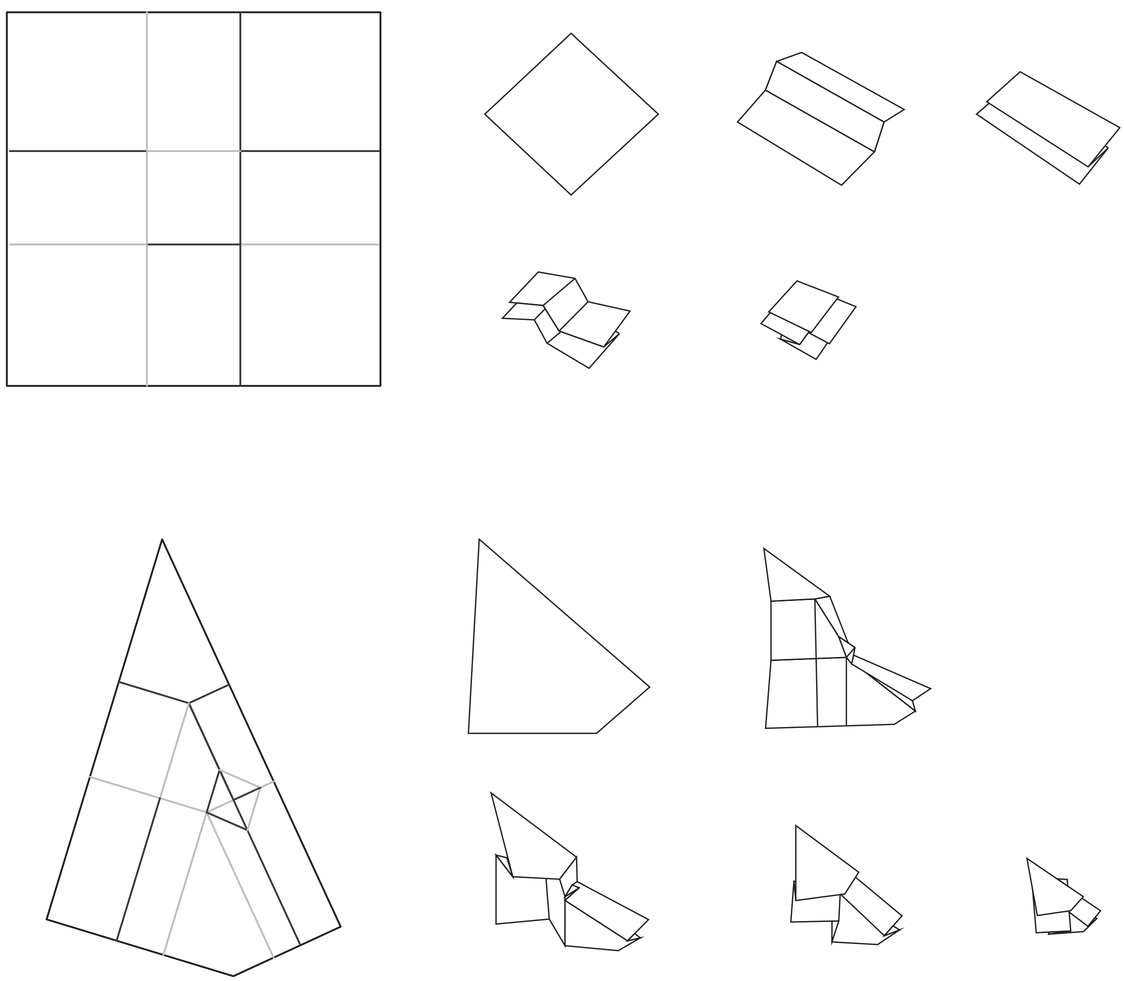

FIGURE 4. Square and quad Double Flip.

The generation procedure of the crease pattern is shown in figure 5. We begin by copying the quad and scaling by $\lambda/2$ about its circumcenter. We then extend lines from each vertex of the inner quad perpendicular to the sides of the parent. We assign mountain and valleys to each of these lines, creating pleats, where the folds on the acute vertex are mountains and the obtuse vertex are valleys. In every case, there are two ways to satisfy Maekawa. Currently the model should fold flat. However, there is an intersection that will occur. One further maneuver is required to guarantee a valid folding. To avoid collision between the two pleats at the obtuse vertex intersection we perform a simple reverse fold so that the pleats can lie inside each other. We reflect line OV about line OV' and extend it till it intersects the long mountain fold CM. This line is then reflected three times around the cross-hair vertex C and assigned mountains and valleys to allow for a valid reverse fold. This reverse fold folds out of the quad, so it cannot intersect any of the flaps inside..

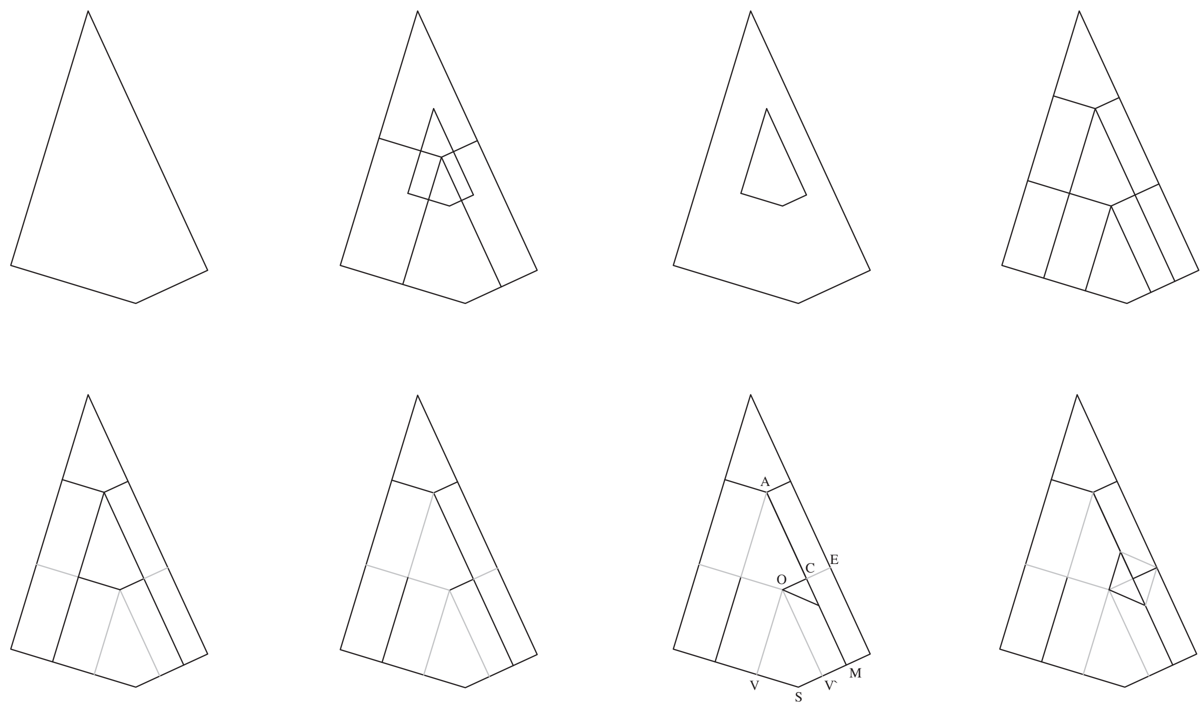

Figure 5. Quad creasing algorithm and reverse fold construction with $\lambda = 1/3$.

It is easy to show that the reverse fold created by the algorithm will be well formed for every possible quad i.e. that the vertices of the reverse fold lie on the correct creases and that the reverse fold does not extend outside of the boundaries of the quad. Using a single pleat the smallest we can scale a quad to is to a third of its size. We will use this worst case scale factor $\lambda = 1/3$ and show that of the vertices of the reverse fold, two lie on OE and two lie on AM. The first vertex of the fold must be O by construction and the other must be E as OC and OE are equal in length. We then observe that extending OV until it intersects AC gives us one of the creases of the reverse fold. OV when extended must always intersects AC as AOV cannot be a line. This is evident as AOS is a line and S and V cannot be the same vertex. The last vertex must now always lie on CM as CM is the same length as AC.

The size of the reverse fold decreases as the quad's acute angle increases and in the case of the square it vanishes. The algorithm will so far fold any quad scaled by λ flat; however the folded form may not be the correct shape as an internal layer may lie outside the boundaries of the shrunken quad. Such overhangs will occur in skinny or lopsided quads and are easily solved by one or more reverse folds. We state the enclosed sandwich lemma to help prove this.

Lemma 1. *If there exists a surface folded flat along a crease, then it consists of a layer t on top and a layer b on the bottom. If a reverse fold is performed to create two new layers m and m' in the middle of t and b such that they fall out of the boundaries of t or b then a finite number of reverse folds can always be carried out on m and m' such that no paper hangs out of the boundaries of t or b.*

Thus if overhangs occur they can be folded back into the model via a reverse fold or sometimes multiple reverse folds, without intersections as the new layers are in between the old ones. In practice there are rarely more than two reverse folds required. Figure 6 shows more examples of the algorithm being applied to quads.

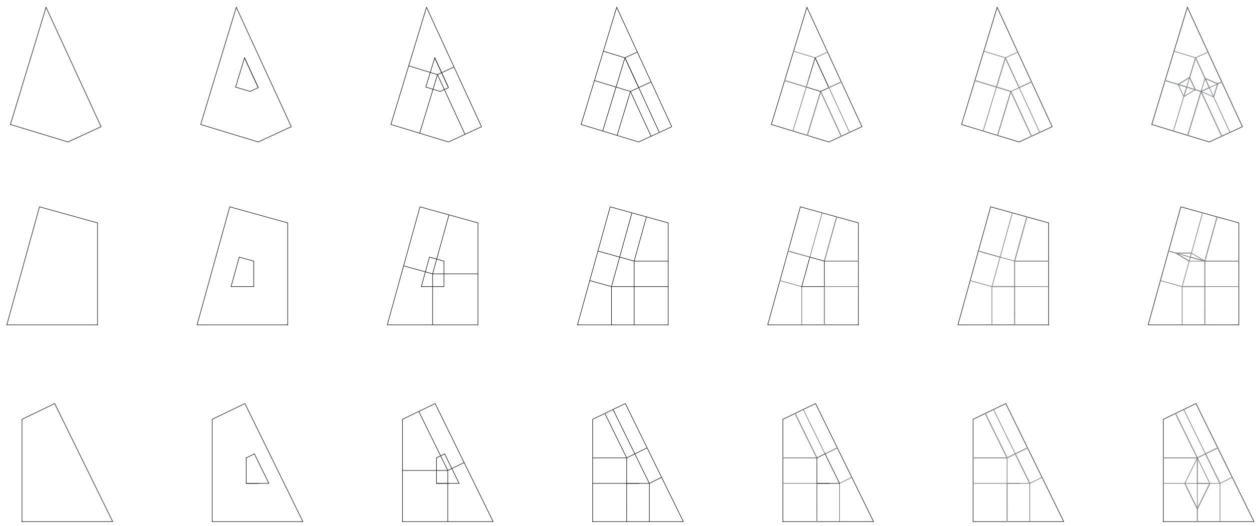

Figure 6. Quad creasing algorithm with $\lambda = 1/2$.

2.5. Merging. It is so far evident that any quad can be scaled between a factor of 1/3 and 1. However, it remains to be seen if this folding holds once the quads are merged. There are two conditions that must hold for a successful merging: the pleats of each quad must meet at the same points and their fold directions must be the same. The former condition must hold by construction as any two quads from one triangle share one side and each inner quad was scaled about the quad circumcenters. The parity condition must also hold because by construction, on the obtuse side of the quad the two pleats had valley folds near the

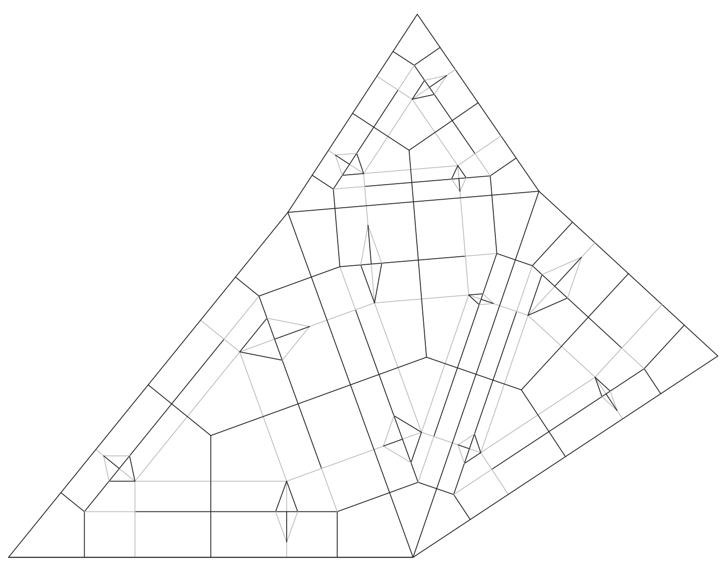

Figure 7. Merging quads into a complete crease pattern.

obtuse vertex and mountain folds away from it. Thus valley folds are always near the center of the triangle and mountain folds surround them. Every acute triangle can be scaled down.

A similar argument follows for merging all triangles. If any two triangles share a side their creases must meet at the same points because the inner quads were scaled about the quad circumcenters. The parity is also correct as there are always two valley folds from the pleats near the middle of each side and two mountain folds surrounding them. In the case where the scale factor is exactly $1/3$, two reverse folds may interface with each other at a point on the side of a triangle. in these cases the models will also fold flat.

Thus all acute triangles can be merged into the original surface. The entire surface can now be folded to scale down to a factor of its size between $1/3$ and 1. If a scale factor less than this is desired the algorithm can simply be re-applied on the scaled surface to further scale it.

3. Acknowledgements

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MIT CSAIL, 32 Vassar St., Cambridge, MA 02139, USA
E-mail address: {edemaine,mdemaine}@mit.edu

McGill University, Montréal, Québec, Canada
E-mail address: kayhan.qaiser@mail.mcgill.ca