A Topologically Convex Vertex-Unfoldable Polyhedron

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Abstract

We construct a polyhedron that is topologically convex (i.e., has the graph of a convex polyhedron) yet has no vertex unfolding: no matter how we cut along the edges and keep faces attached at vertices to form a connected (hinged) surface, the surface necessarily unfolds with overlap.

1 Introduction

Polyhedron unfolding has a long history dating back to Albrecht Dürer in 1525; see [3]. In general, the goal is to cut along a one-dimensional subset of the polyhedron's surface to enable the remainder of the surface to unfold into the plane without overlap. An *edge unfolding* consists of cutting along a subset of the edges of the polyhedron, while keeping the surface interior-connected; the planar unfolding is then uniquely determined by the development (local unfolding) of the intrinsic metric in the plane. A *vertex unfolding* consists of cutting along a subset of the edges, typically all of them, while keeping the faces connected together via shared vertices (without any crossing connections at the vertices); the planar unfolding is no longer unique, but rather acts like a hinged dissection, with faces able to rotate around shared vertex hinges.

Vertex unfolding was introduced in [2] as a less restrictive form of edge unfolding. They proved that every triangulated manifold (in any dimension, though we focus here on 2-manifolds in 3D) has a vertex unfolding. This result shows that vertex unfolding is more powerful than edge unfolding, as there are triangulated polyhedra that are *edge-ununfoldable* (have no edge unfolding) [1].

In this paper, we solve the "obvious question left open" by [2]: to what extent is the assumption of triangular faces necessary for vertex unfolding? Specifically, they asked whether every polyhedron with simply connected faces has a vertex unfolding, and whether every polyhedron with convex faces has a vertex unfolding.

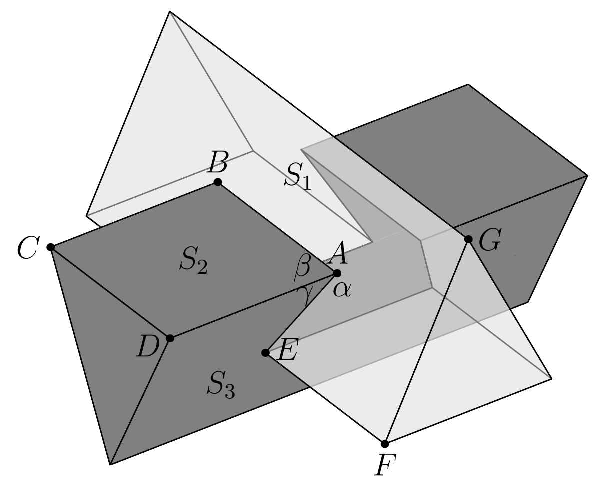

Figure 1: The polyhedron P is a union of two identical overlapping triangular prisms, and—with proper dimension choices—has no vertex unfolding. The labeled points have coordinates $A = (1, 2, 1)$, $B = (1, -2, 1)$, $C = (5, -2, 1)$, $D = (5, 2, 1)$, $E = (2, 1, -1)$, $F = (2, 5, -1)$, $G = (0, 5, 3)$; the rest can be derived from symmetries around the z-axis and the lines $x = \pm y$ in the xy-plane.

We prove that the answer to the first problem is "no", though the second problem remains open.

More precisely, we construct "topologically convex" vertex-ununfoldable polyhedra, strengthening the CCCG 1999 result of edge-ununfoldable polyhedra [1]. A polyhedron is *topologically convex* if its graph (1-skeleton) is the graph of a convex polyhedron, or equivalently by Steinitz's Theorem, it is 3-connected and planar. In terms of the polyhedron's surface, topological convexity is equivalent to requiring that every face is homeomorphic to a disk (as they are in a convex polyhedron), and that every two faces meet at one edge, one vertex, or not at all (as they would in a convex polyhedron). In particular, topological convexity forbids the example of a small box attached in the middle of a face of another box, which is the only previously known vertex-unfoldable polyhedron [2].

2 Vertex-Ununfoldable Polyhedra

We present two related topologically convex vertex-ununfoldable polyhedra. Our first example, P, is sim-

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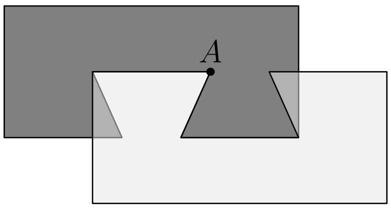

Figure 2: If faces S_1 and S_3 were hinged at A, they must be in this configuration by Observation 2. But as there are overlaps, this is not allowed.

ply the union of two overlapping, identical triangular prisms, as shown in Figure 1. For concreteness's sake, we have listed coordinates of the labeled vertices, and the rest can be inferred from symmetries. To prove that P has no vertex unfolding, we make two self-evident observations:

Observation 1 *If two polygons T_1 and T_2 have two vertices $v_1 \in T_1$ and $v_2 \in T_2$ whose angles add to more than $360°$, then these vertices cannot be hinged in the plane without the polygons overlapping.*

Observation 2 *If the angles at v_1 and v_2 add to exactly $360°$, and if these vertices are hinged without overlap in the plane, then they must be oriented to exactly cover the $360°$ surrounding the hinge.*

Notice these obstructions to vertex unfoldings are entirely local in nature, involving only two polygons joined at a vertex.[1] These observations alone are enough to prove our claim:

Theorem 3 *Polyhedron P has no vertex unfolding.*

Proof. We will show that no lightly shaded face (as in Figure 1) can connect to a dark face in any planar vertex hinging of the faces, and therefore any proposed vertex unfolding is disconnected. Indeed, any light-dark connection must happen at one of the eight central vertices, and as they are all identical under symmetry, we may focus on vertex A. Because $\alpha > 270°$ and $\beta = 90°$, Observation 1 implies that S_1 and S_2 cannot hinge at A. Because $\alpha + \gamma = 360°$, by Observation 2, if S_1 and S_3 were hinged at A then they must be hinged as in Figure 2. But the dimensions were chosen so that these polygons would overlap in this configuration. □

By contrast, if the unfolding is allowed to have two connected components, then an *edge* unfolding is possible, as in Figure 3. Also, the use of Observation 2 required more global knowledge than just the vertex angles: the shapes of polygons S_1 and S_3 were crucial. Indeed, if AD (and all symmetric copies) were chosen shorter, then an edge unfolding of P would be possible, as in Figure 4.

[1] For the related edge-unfolding problem, these are called ***1-local*** obstructions [4].

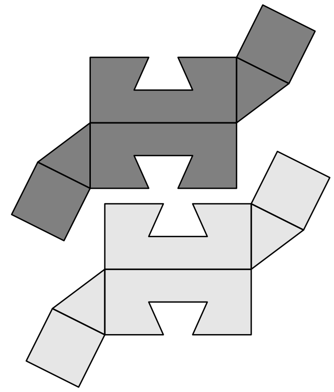

Figure 3: An edge unfolding of P into two connected components.

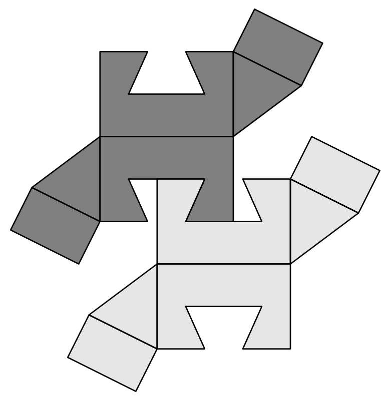

Figure 4: If the prisms were shorter, an edge unfolding would exist, as depicted here.

In fact, Observation 1 alone is sufficient to provide a vertex-ununfoldable polyhedron. Perturb polyhedron P to a new polyhedron P' by increasing γ slightly (while maintaining symmetry) so that $\alpha + \gamma' > 360°$; this also increases β slightly to β'. Such a polyhedron P' is shown in Figure 5. Because $\alpha + \beta', \alpha + \gamma' > 360°$, it follows by Observation 1 that S_1' cannot hinge to S_2' or S_3' at vertex A, so as before, any vertex unfolding must be disconnected:

Theorem 4 *Polyhedron P' has no vertex unfolding.*

3 Open Questions

The foremost open question concerning vertex unfolding is to find the largest natural class of polyhedra that always admit vertex unfoldings. We have shown here that topologically convex is too large a class. In fact, topologically convex and star shaped is too large, because both P and P' are star shaped—in both cases, the origin can see the entire polyhedron.

Another natural question, posed in [2], is whether (topologically convex) polyhedra with convex faces admit vertex unfoldings. We conjecture that the answer

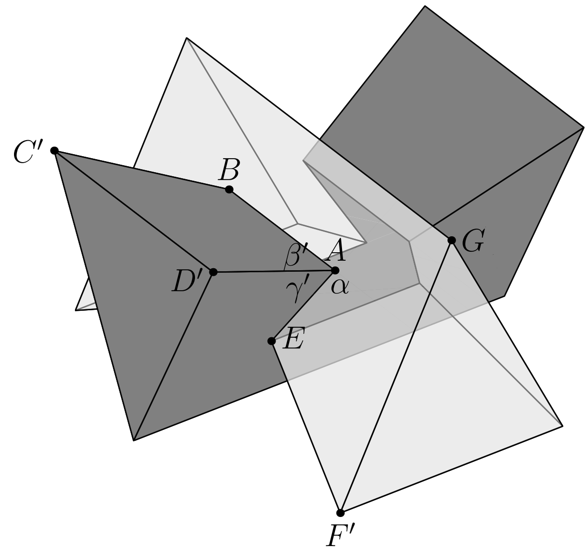

Figure 5: Polyhedron P' is obtained from P by moving vertex C to $C' = (5, -3, 3)$ and similarly for its symmetric copies. This polyhedron has no vertex unfolding based solely on the fact that $\alpha + \beta' > 360°$ and $\alpha + \gamma' > 360°$.

is "no", but the methods used in this paper cannot be directly extended. Indeed, any vertex of such a polyhedron with negative curvature must have at least four incident faces, any two of which could potentially remain connected, so the local conclusions are not as strong.

Finally, we echo an open problem implicit in [2] and explicit in [3, Open Problem 22.20]: does every convex polyhedron have a vertex unfolding? This is a weaker form of the famous convex edge-unfolding conjecture [3].

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